

ON SOME BANACH SEQUENCE SPACES DERIVED BY A NEW BAND MATRIX

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ABSTRACT. In this paper, we define the band matrix $T = (t_{nk})$ by

$$t_{nk} = \begin{cases} t_n & , \quad k = n \\ -\frac{1}{t_n} & , \quad k = n - 1 \\ 0 & , \quad k > n \text{ or } 0 \leq k < n - 1, \end{cases}$$

where $t_n > 0$ for all $n \in \mathbb{N}$ and $(t_n) \in c \setminus c_0$. By using the matrix T , we introduce the sequence space $\ell_p(T)$ for $1 \leq p \leq \infty$. Also, we give some inclusion theorems related to this space and find the α -, β -, γ -duals of the space $\ell_p(T)$. Lastly, we investigate the necessary and sufficient conditions for an infinite matrix to be in the classes $(\ell_p(T), \lambda)$ or $(\lambda, \ell_p(T))$ and give the norm of the operators in $B(\ell_p(T), \mu(S))$, where $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$ and $\mu \in \{\ell_1, \ell_\infty\}$.

1. Introduction and Preliminaries

Let ω be the space of all real or complex valued sequences. We shall write $\sup_k$ and $\sum_k$ instead of $\sup_{k \in \mathbb{N}}$ and $\sum_{k=0}^\infty$, respectively, where $\mathbb{N} = \{0, 1, 2, \dots\}$. Also, if $x = (x_k)_{k=0}^\infty \in \omega$, we simply denote it by $x = (x_k)$. Further, $e = (1, 1, \dots)$ and $e^{(k)}$ is the sequence whose k th term is 1 and the other terms are 0, that is, $e^{(k)} = (e_0^{(k)}, e_1^{(k)}, \dots, e_k^{(k)}, \dots) = (0, 0, \dots, 1, \dots)$. Any vector subspace of ω is called a *sequence space*. By ℓ_∞, c, c_0 and ℓ_p ($1 \leq p < \infty$), we denote the spaces of all bounded, convergent, null sequences and p -absolutely convergent series, respectively.

A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_n : \lambda \rightarrow \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$, where $\mathbb{C}$ is the set of all complex numbers. If a *K-space* λ is a complete linear metric space, it is called an *FK-space*. A normed *FK-space* is called a *BK-space*, that is, a *BK-space* is a Banach sequence space. For example, the sequence space ℓ_∞ is a *BK-space* with the norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$. Further, ℓ_p is a complete p -normed space and a *BK-space* in the cases of $0 < p < 1$ and $1 \leq p < \infty$ with respect to the usual p -norm and ℓ_p -norm defined by

$$\|x\|_{\ell_p} = \sum_k |x_k|^p \quad (0 < p < 1)$$

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and

$$\|x\|_{\ell_p} = \left(\sum_k |x_k|^p \right)^{1/p} \quad (1 \leq p < \infty),$$

respectively.

Let λ and μ be sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} , where $n, k \in \mathbb{N}$. Then A gives a matrix transformation from λ into μ and we write $A : \lambda \rightarrow \mu$ if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = (A_n(x))$, the A -transform of x , is in μ , where

$$A_n(x) = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \quad (1.1)$$

By (λ, μ) , we denote the class of all infinite matrices that map λ into μ . Hence $A \in (\lambda, \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and $Ax \in \mu$ for all $x \in \lambda$. If λ and μ are any two Banach spaces, then $B(\lambda, \mu)$ denotes the set of all bounded linear operators from λ into μ . The operator norm of $A \in B(\lambda, \mu)$ is given by $\|A\| = \sup\{\|Ax\|_\mu : x \in \lambda, \|x\|_\lambda \leq 1\}$.

Let λ be a sequence space. Then the *matrix domain* λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}$$

which is also a sequence space.

In the literature, there are many papers related to new sequence spaces constructed by means of the matrix domain of a special triangle. See, for example [1–8]. For more details about matrix domains of triangles, one can see [9].

A sequence (b_n) in normed space λ is called a *Schauder basis* for λ if for every $x \in \lambda$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e.,

$$\lim_{m \rightarrow \infty} \|x - \sum_{n=0}^m \alpha_n b_n\| = 0.$$

By cs_0 , cs and bs , we denote the set of all convergent to zero, convergent and bounded series, respectively, that is, $cs_0 = \{x = (x_k) \in \omega : (\sum_{k=0}^n x_k)_{n=0}^\infty \in c_0\}$, $cs = \{x = (x_k) \in \omega : (\sum_{k=0}^n x_k)_{n=0}^\infty \in c\}$ and $bs = \{x = (x_k) \in \omega : (\sum_{k=0}^n x_k)_{n=0}^\infty \in \ell_\infty\}$, and we define the norm on cs_0 , cs and bs by $\|x\|_{cs_0} = \|x\|_{cs} = \|x\|_{bs} = \sup_n |\sum_{k=0}^n x_k|$. Let λ and μ be subsets of ω . For all $z \in \omega$, we write $z^{-1} * \mu = \{x \in \omega : xz = (x_k z_k) \in \mu\}$. The set $Z = M(\lambda, \mu) = \cap_{x \in \lambda} x^{-1} * \mu = \{a \in \omega : ax \in \mu \text{ for all } x \in \lambda\}$ is called the multiplier space of λ and μ . In the special case, where $\mu = \ell_1$, $\mu = cs$ or $\mu = bs$, the multiplier spaces $\lambda^\alpha = M(\lambda, \ell_1)$, $\lambda^\beta = M(\lambda, cs)$ and $\lambda^\gamma = M(\lambda, bs)$ are called the α -, β - and γ -duals of λ .

Throughout this paper, we assume that $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

The difference operator $\Delta : \omega \rightarrow \omega$ is defined by $\Delta x = (\Delta x_k) = (x_k - x_{k-1})$ or $\Delta x = (\Delta x_k) = (x_{k-1} - x_k)$ for all $x = (x_k) \in \omega$. The matrix domain λ_Δ is called the difference sequence space whenever λ is a sequence space. Firstly, the notion of difference sequence spaces was defined by Kızmaz [10] as

$$\lambda(\Delta) = \{x = (x_k) \in \omega : (x_k - x_{k-1}) \in \lambda\}$$

for $\lambda = \ell_\infty, c$ and c_0 . After Kızmaz, Et and Çolak [11] defined the generalized difference sequence spaces

$$\begin{aligned}\ell_\infty(\Delta^m) &= \{x = (x_k) \in \omega : \Delta^m x \in \ell_\infty\}, \\ c(\Delta^m) &= \{x = (x_k) \in \omega : \Delta^m x \in c\}\end{aligned}$$

and

$$c_0(\Delta^m) = \{x = (x_k) \in \omega : \Delta^m x \in c_0\},$$

where $m \in \mathbb{N}$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and so that

$$\Delta^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k+i}.$$

The difference space

$$bv_p = \{x = (x_k) \in \omega : (x_k - x_{k-1}) \in \ell_p\} \quad (0 < p < \infty)$$

was studied by Altay and Başar [12] for $0 < p < 1$ and in the case $1 \leq p \leq \infty$ Başar and Altay [13], and Çolak et al [14]. Recently, for $\lambda \in \{\ell_p, c_0, c, \ell_\infty\}$ ($1 \leq p < \infty$), Kirişçi and Başar [6] introduced the generalized difference sequence space

$$\widehat{\lambda} = \{x = (x_k) : \in \omega : B(r, s)x = ((B(r, s)x)_k) \in \lambda\},$$

where $B(r, s)x$ is the sequence defined by $(B(r, s)x)_k = rx_k + sx_{k-1}$ for all $k \in \mathbb{N}$ and $r, s \in \mathbb{R} \setminus \{0\}$. Quite recently, the sequence space

$$\lambda(B) = \{x = (x_k) \in \omega : B(r, s, t)x = ((B(r, s, t)x)_k) \in \lambda\}$$

was studied by Sönmez [15], where $(B(r, s, t)x)_k = rx_k + sx_{k-1} + tx_{k-2}$ for all $k \in \mathbb{N}$ and $r, s, t \in \mathbb{R} \setminus \{0\}$.

In [16], the Fibonacci band matrix $\widehat{F} = (\widehat{f}_{nk})$ was defined by

$$\widehat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n} & , \quad k = n - 1 \\ \frac{f_n}{f_{n+1}} & , \quad k = n \\ 0 & , \quad 0 \leq k < n - 1 \text{ or } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$, where f_n is the n th Fibonacci number ($n \in \mathbb{N}$). Also, in [16] the Fibonacci difference sequence spaces introduced as follows:

$$\ell_p(\widehat{F}) = \left\{ x = (x_n) \in \omega : \sum_n \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^p < \infty \right\} \quad (1 \leq p < \infty)$$

and

$$\ell_\infty(\widehat{F}) = \left\{ x = (x_n) \in \omega : \sup_n \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right| < \infty \right\},$$

where $\widehat{F}x$ is the $\widehat{F}$ -transform of a sequence $x = (x_n)$.

Candan [17] defined the sequential generalized difference matrix $B(\tilde{r}, \tilde{s}) = \{b_{nk}(\tilde{r}, \tilde{s})\}$ by

$$b_{nk}(\tilde{r}, \tilde{s}) = \begin{cases} r_n & , \quad k = n \\ s_n & , \quad k = n - 1 \\ 0 & , \quad 0 \leq k < n - 1 \text{ or } k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, where $\tilde{r} = (r_n)$ and $\tilde{s} = (s_n)$ are convergent sequences of positive real numbers. Moreover, Candan [17] introduced the sequence space

$$\tilde{\lambda} = \{x = (x_k) \in \omega : B(\tilde{r}, \tilde{s})x = ((B(\tilde{r}, \tilde{s})x)_k) \in \lambda\},$$

where $(B(\tilde{r}, \tilde{s})x)_k = s_{k-1}x_{k-1} + r_kx_k$ for all $k \in \mathbb{N}$, $\lambda \in \{\ell_\infty, c, c_0, \ell_p\}$ and $1 \leq p < \infty$. Further in [18–25], several authors defined and studied some new difference sequence spaces.

The paper is organized so that this section is followed by three sections. In Section 2 we define a new band matrix and introduce the sequence spaces $\ell_p(T)$ and $\ell_\infty(T)$, where $1 \leq p < \infty$. We prove that $\ell_p(T)$ and $\ell_\infty(T)$ are Banach spaces with respect to the norm defined on these spaces. Further, we establish some inclusion theorems related to the space $\ell_p(T)$, where $1 \leq p \leq \infty$. In section 3 we determine the α -, β -, γ - duals of the space $\ell_p(T)$ for $1 \leq p \leq \infty$. In the last section we characterize the classes $(\ell_p(T), \lambda)$ and $(\lambda, \ell_p(T))$ for $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$ and also give the norm of an operator in the class $(\ell_p(T), \mu(S))$ for $\mu \in \{\ell_1, \ell_\infty\}$, where S is the band matrix having the same properties with T and $1 \leq p \leq \infty$.

2. The difference sequence space $\ell_p(T)$

In the present section, we define a new band matrix T and introduce the difference sequence space $\ell_p(T)$ by using this matrix, where $1 \leq p \leq \infty$. Also, we present some theorems which give inclusion relations concerning this space.

Let $t_n > 0$ for all $n \in \mathbb{N}$ and $t = (t_n) \in c \setminus c_0$. We define the band matrix $T = (t_{nk})$ by

$$t_{nk} = \begin{cases} t_n & , \quad k = n \\ -\frac{1}{t_n} & , \quad k = n - 1 \\ 0 & , \quad k > n \text{ or } 0 \leq k < n - 1. \end{cases}$$

One can easily derive that the inverse $T^{-1} = (g_{nk})$ of the band matrix T is given by

$$g_{nk} = \begin{cases} t_k \prod_{j=k}^n \frac{1}{t_j^2} & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$.

Now, we introduce the difference sequence spaces $\ell_p(T)$ and $\ell_\infty(T)$ by

$$\ell_p(T) = \left\{ x = (x_n) \in \omega : \sum_n \left| t_n x_n - \frac{1}{t_n} x_{n-1} \right|^p < \infty \right\} \quad (1 \leq p < \infty)$$

and

$$\ell_\infty(T) = \left\{ x = (x_n) \in \omega : \sup_n \left| t_n x_n - \frac{1}{t_n} x_{n-1} \right| < \infty \right\}.$$

As the notation of matrix domain, the sequence spaces $\ell_p(T)$ and $\ell_\infty(T)$ may be **represented** by

$$\ell_p(T) = (\ell_p)_T \quad (1 \leq p < \infty) \quad \text{and} \quad \ell_\infty(T) = (\ell_\infty)_T.$$

The space $\ell_p(T)$ is more general and more comprehensive than the spaces $\ell_p(\widehat{F})$ and bv_p , that is $\ell_p(T)$ contains both of them, where $1 \leq p \leq \infty$. Let $t_n = \frac{f_n}{f_{n+1}}$ ($n \in \mathbb{N}$), then T is the Fibonacci band matrix $\widehat{F}$ and let $(t_n) = e$, then T is the difference matrix Δ . On the other hand, the space $\ell_p(T)$ is not a special case of the space $\widetilde{\ell}_p = (\ell_p)_{B(\widetilde{r}, \widetilde{s})}$ defined by Candan [17]. To put it more explicitly, if we take t_k and $-1/t_k$ instead of r_k and s_{k-1} , respectively, this contradicts the fact that $s_k > 0$ for all $k \in \mathbb{N}$.

We will frequently use the sequence $y = (y_n)$ for the T -transform of a sequence $x = (x_n)$, that is,

$$y_n = T_n(x) = \begin{cases} t_0 x_0 & , \quad n = 0 \\ t_n x_n - \frac{1}{t_n} x_{n-1} & , \quad n \geq 1 \end{cases} \quad (n \in \mathbb{N}). \quad (2.1)$$

Theorem 2.1. *Let $1 \leq p \leq \infty$. Then $\ell_p(T)$ is a Banach space with the norm $\|x\|_{\ell_p(T)} = \|Tx\|_{\ell_p}$, that is,*

$$\|x\|_{\ell_p(T)} = \begin{cases} \left(\sum_n |T_n(x)|^p \right)^{1/p} & , \quad 1 \leq p < \infty \\ \sup_n |T_n(x)| & , \quad p = \infty. \end{cases}$$

Proof. Suppose that $\|x\|_{\ell_p(T)} = 0$. Then, $\|Tx\|_{\ell_p} = 0$ and since $\|\cdot\|_{\ell_p}$ is a norm we have $Tx = \theta$. Since T is invertible, we have $x = \theta$.

Let $\alpha \in \mathbb{C}$ and $x \in \ell_p(T)$. Then,

$$\begin{aligned} \|\alpha x\|_{\ell_p(T)} &= \|T(\alpha x)\|_{\ell_p} = \|\alpha Tx\|_{\ell_p} \\ &= |\alpha| \|Tx\|_{\ell_p} = |\alpha| \|x\|_{\ell_p(T)}. \end{aligned}$$

Lastly, for $x, z \in \ell_p(T)$ we have

$$\begin{aligned} \|x + z\|_{\ell_p(T)} &= \|T(x + z)\|_{\ell_p} = \|Tx + Tz\|_{\ell_p} \\ &\leq \|Tx\|_{\ell_p} + \|Tz\|_{\ell_p} = \|x\|_{\ell_p(T)} + \|z\|_{\ell_p(T)} \end{aligned}$$

and so the triangle inequality holds.

Hence, $(\ell_p(T), \|\cdot\|_{\ell_p(T)})$ is a normed sequence space for $1 \leq p \leq \infty$. To prove that $\ell_p(T)$ is a Banach space, let (x_n) be a Cauchy sequece in $\ell_p(T)$. Then, (y_n) is a sequence in ℓ_p . Clearly,

$$\begin{aligned} \|x_n - x_m\|_{\ell_p(T)} &= \|T(x_n - x_m)\|_{\ell_p} \\ &= \|Tx_n - Tx_m\|_{\ell_p} = \|y_n - y_m\|_{\ell_p}, \end{aligned}$$

that is, (y_n) is a Cauchy sequence in ℓ_p . Since $(\ell_p, \|\cdot\|_{\ell_p})$ is a Banach space, there exists $y \in \ell_p$ such that $\lim_{n \rightarrow \infty} y_n = y$ in ℓ_p . Since $x = T^{-1}y$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - x\|_{\ell_p(T)} &= \lim_{n \rightarrow \infty} \|T(x_n - x)\|_{\ell_p} \\ &= \lim_{n \rightarrow \infty} \|Tx_n - Tx\|_{\ell_p} = \lim_{n \rightarrow \infty} \|y_n - y\|_{\ell_p} = 0. \end{aligned}$$

This means that $\lim_{n \rightarrow \infty} x_n = x$ in $\ell_p(T)$, where $x \in \ell_p(T)$. The proof is completed.

Remark 2.2. *It is clear that $\ell_p(T)$ is a BK-space for $1 \leq p \leq \infty$.*

Theorem 2.3. *The difference sequence space $\ell_p(T)$ is linearly isomorphic to the space ℓ_p , that is, $\ell_p(T) \cong \ell_p$ for $1 \leq p \leq \infty$.*

Proof. We must show that there exists a linear bijection between the spaces $\ell_p(T)$ and ℓ_p for $1 \leq p \leq \infty$. Let T be the transformation defined from $\ell_p(T)$ to ℓ_p by $x \rightarrow y = Tx = (T_n(x))$. Then, we have $Tx = y \in \ell_p$ for every $x \in \ell_p(T)$. It is clear that T is a linear transformation. Also, it is obvious that $x = \theta$ whenever $Tx = \theta$ and so that T is injective.

Furthermore, let $y = (y_n) \in \ell_p$ be given for $1 \leq p \leq \infty$ and define the sequence $x = (x_n)$ as follows:

$$x_n = \sum_{k=0}^n \left(\prod_{j=k}^n \frac{1}{t_j^2} \right) t_k y_k \quad (n \in \mathbb{N}). \quad (2.2)$$

Then, by combining (2.1) and (2.2), we get for every $n \in \mathbb{N}$

$$\begin{aligned} T_n(x) &= t_n x_n - \frac{1}{t_n} x_{n-1} \\ &= t_n \sum_{k=0}^n \left(\prod_{j=k}^n \frac{1}{t_j^2} \right) t_k y_k - \frac{1}{t_n} \sum_{k=0}^{n-1} \left(\prod_{j=k}^{n-1} \frac{1}{t_j^2} \right) t_k y_k \\ &= t_n \left(\prod_{j=n}^n \frac{1}{t_j^2} \right) t_n y_n + t_n \sum_{k=0}^{n-1} \left(\prod_{j=k}^n \frac{1}{t_j^2} \right) t_k y_k - \frac{1}{t_n} \sum_{k=0}^{n-1} \left(\prod_{j=k}^{n-1} \frac{1}{t_j^2} \right) t_k y_k \\ &= y_n. \end{aligned}$$

This means that $Tx = y$. Since $y \in \ell_p$, we have $Tx \in \ell_p$. Thus, we conclude that $x \in \ell_p(T)$ for any $y \in \ell_p$. Hence T is surjective.

Since $\|x\|_{\ell_p(T)} = \|Tx\|_{\ell_p}$ for any $x \in \ell_p(T)$, we have

$$\|y\|_{\ell_p} = \|Tx\|_{\ell_p} = \|x\|_{\ell_p(T)}$$

which shows that T preserves the norm, where $1 \leq p \leq \infty$. Hence, T is an isometry. As a consequence, the spaces $\ell_p(T)$ and ℓ_p are isometrically isomorphic for $1 \leq p \leq \infty$. This completes the proof.

It is known that the space ℓ_p is not a Hilbert space with $p \neq 2$. The similar result is valid for the space $\ell_p(T)$ and the following theorem gives this result.

Theorem 2.4. *The space $\ell_p(T)$ is not an inner product space in the case $p \neq 2$. Hence, $\ell_p(T)$ is not a Hilbert space for $1 \leq p < \infty$ and $p \neq 2$.*

Proof. We must show that the space $\ell_2(T)$ is a Hilbert space while $\ell_p(T)$ is not a Hilbert space for $p \neq 2$. By Theorem 2.1, we know that $\ell_2(T)$ is a Banach space with the norm $\|x\|_{\ell_2(T)} = \|Tx\|_{\ell_2}$ and its norm can be obtained as follows:

$$\|x\|_{\ell_2(T)} = \langle x, x \rangle_{\ell_2(T)}^{1/2} = \langle Tx, Tx \rangle_{\ell_2}^{1/2} = \|Tx\|_{\ell_2}$$

for every $x \in \ell_2(T)$. Hence $\ell_2(T)$ is a Hilbert space.

Consider the sequences

$$u = (u_n) = \begin{cases} \frac{1}{t_0} & , \quad n = 0 \\ (t_1 + \frac{1}{t_0}) \prod_{i=1}^n \frac{1}{t_i^2} & , \quad n \geq 1 \end{cases} \quad (n \in \mathbb{N})$$

and

$$v = (v_n) = \begin{cases} \frac{1}{t_0} & , \quad n = 0 \\ (-t_1 + \frac{1}{t_0}) \prod_{i=1}^n \frac{1}{t_i^2} & , \quad n \geq 1 \end{cases} \quad (n \in \mathbb{N}).$$

With the T -transforms of u and v , we have the following sequences

$$Tu = (1, 1, 0, 0, \dots) \text{ and } Tv = (1, -1, 0, 0, \dots).$$

Also, it is easy to see that

$$\|u + v\|_{\ell_p(T)}^2 + \|u - v\|_{\ell_p(T)}^2 = 8 \neq 4(2^{2/p}) = 2(\|u\|_{\ell_p(T)}^2 + \|v\|_{\ell_p(T)}^2)$$

for $p \neq 2$. This means that the parallelogram equality cannot be satisfied by the norm of the space $\ell_p(T)$ for $p \neq 2$. Therefore, this norm cannot be obtained from an inner product. Hence, the space $\ell_p(T)$ with $p \neq 2$ is a Banach space but it is not a Hilbert space, where $1 \leq p < \infty$. The proof is completed.

Remark 2.5. *Obviously, the space $\ell_\infty(T)$ is also a Banach space but it is not a Hilbert space.*

Before giving some inclusion relations concerning the space $\ell_p(T)$, we give a lemma which is necessary to show that some inclusions strictly hold.

Lemma 2.6. *[26, Theorem 3. page 219] A product $\prod_n (1 + a_n)$ with positive terms a_n is convergent if and only if the series $\sum_n a_n$ converges.*

Now, we present the inclusion relations concerning the space $\ell_p(T)$.

Theorem 2.7. *The inclusion $\ell_p(T) \subset \ell_q(T)$ strictly holds for $1 \leq p < q < \infty$.*

Proof. Let $1 \leq p < q < \infty$. If x is any sequence in $\ell_p(T)$, then its T -transform Tx is in ℓ_p . Since the inclusion $\ell_p \subset \ell_q$ holds, Tx is also in ℓ_q . Hence $x \in \ell_q(T)$ which means that $\ell_p(T) \subset \ell_q(T)$. To show that the inclusion is strict, consider a sequence $x = (x_n) \in \ell_q$ but not in ℓ_p , i.e.,

$x \in \ell_q \setminus \ell_p$. From the fact that the inclusion $\ell_p \subset \ell_q$ is strict, the existence of $x \in \ell_q \setminus \ell_p$ is clear. Let define the sequence $y = (y_n)$ in terms of the sequence x as follows:

$$y_n = \sum_{k=0}^n \left(\prod_{j=k}^n \frac{1}{t_j^2} \right) t_k x_k \quad (n \in \mathbb{N}).$$

Then, it is clear that

$$T_n(y) = x_n$$

for every $n \in \mathbb{N}$. This shows that $Ty = x$ and since $x \in \ell_q \setminus \ell_p$, we have $Ty \in \ell_q \setminus \ell_p$. Hence, the sequence y must be in $\ell_q(T)$ but cannot be in $\ell_p(T)$, that is, the inclusion $\ell_p(T) \subset \ell_q(T)$ is strict. The proof is completed.

Theorem 2.8. *The inclusion $\ell_p(T) \subset \ell_\infty(T)$ strictly holds for $1 \leq p < \infty$.*

Proof. If $x \in \ell_p(T)$, then $Tx \in \ell_p$. Since $\ell_p \subset \ell_\infty$, $Tx \in \ell_\infty$. Hence, $x \in \ell_\infty(T)$ which shows that $\ell_p(T) \subset \ell_\infty(T)$. To show that this inclusion is strict, we define the sequence $y = (y_n)$ by

$$y_n = \sum_{k=0}^n (-1)^k t_k \prod_{i=k}^n \frac{1}{t_i^2} \quad (n \in \mathbb{N}).$$

Then, we have for every $n \in \mathbb{N}$ that

$$\begin{aligned} T_n(y) &= t_n \sum_{k=0}^n (-1)^k t_k \prod_{i=k}^n \frac{1}{t_i^2} - \frac{1}{t_n} \sum_{k=0}^{n-1} (-1)^k t_k \prod_{i=k}^{n-1} \frac{1}{t_i^2} \\ &= t_n (-1)^n t_n \prod_{i=n}^n \frac{1}{t_i^2} + t_n \sum_{k=0}^{n-1} (-1)^k t_k \prod_{i=k}^n \frac{1}{t_i^2} - \frac{1}{t_n} \sum_{k=0}^{n-1} (-1)^k t_k \prod_{i=k}^{n-1} \frac{1}{t_i^2} \\ &= (-1)^n + \sum_{k=0}^{n-1} (-1)^k t_k \left(t_n \prod_{i=k}^n \frac{1}{t_i^2} - \frac{1}{t_n} \prod_{i=k}^{n-1} \frac{1}{t_i^2} \right) \\ &= (-1)^n. \end{aligned}$$

Then, $Ty \in \ell_\infty \setminus \ell_p$ since $((-1)^n) \in \ell_\infty$ but not in ℓ_p . Thus, y is in $\ell_\infty(T)$ but not in $\ell_p(T)$ which means that the inclusion $\ell_p(T) \subset \ell_\infty(T)$ strictly holds. The proof is completed.

Theorem 2.9. *The inclusion $\ell_p \subset \ell_p(T)$ strictly holds, where $1 \leq p \leq \infty$.*

Proof. To prove that the inclusion $\ell_p \subset \ell_p(T)$ holds for $1 \leq p \leq \infty$, it is sufficient to show that there exists a number $M > 0$ such that $\|x\|_{\ell_p(T)} \leq M \|x\|_{\ell_p}$ for any $x \in \ell_p$.

Let $x \in \ell_p$ and $1 \leq p \leq \infty$. $\left(\frac{1}{t_n}\right) \in c \setminus c_0$ since $(t_n) \in c \setminus c_0$. Then, there exist $K, L > 0$ such that $t_n \leq K$ and $\frac{1}{t_n} \leq L$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \|x\|_{\ell_p(T)} &= \left(\sum_n |T_n(x)|^p \right)^{1/p} \\ &= \left(\sum_n \left| t_n x_n - \frac{1}{t_n} x_{n-1} \right|^p \right)^{1/p} \\ &\leq \left(\sum_n |t_n x_n|^p \right)^{1/p} + \left(\sum_n \left| \frac{1}{t_n} x_{n-1} \right|^p \right)^{1/p} \\ &\leq \left(K^p \sum_n |x_n|^p \right)^{1/p} + \left(L^p \sum_n |x_{n-1}|^p \right)^{1/p} \\ &= (K + L) \|x\|_{\ell_p} \end{aligned}$$

and

$$\begin{aligned} \|x\|_{\ell_\infty(T)} &= \sup_n |T_n(x)| = \sup_n \left| t_n x_n - \frac{1}{t_n} x_{n-1} \right| \\ &\leq (K + L) \sup_n |x_n| = (K + L) \|x\|_{\ell_\infty}. \end{aligned}$$

If we define $M = K + L$, it yields the desired result, that is, $\|x\|_{\ell_p(T)} \leq M \|x\|_{\ell_p}$ for $1 \leq p \leq \infty$. To prove that the inclusion is strict:

i) If $0 < t_n < 1$ for all $n \in \mathbb{N}$;

For $1 \leq p < \infty$, the sequence $x = \left(\prod_{i=1}^n \frac{1}{t_i^2} \right) \notin \ell_p$ since $1/t_i > 1$. Clearly, $Tx = (t_0, 0, 0, \dots) \in \ell_p$. Thus $x \in \ell_p(T)$.

Let $t_i = \sqrt{\frac{i}{i+1}} < 1$ for all $i \in \mathbb{N}$. Then, $\frac{1}{t_i^2} = 1 + \frac{1}{i}$. By Lemma 2.6, $\prod_{i=1}^\infty \frac{1}{t_i^2}$ is not convergent since $\sum_{i=1}^\infty \frac{1}{i}$ is not convergent. Hence, $x = \left(\prod_{i=1}^n \frac{1}{t_i^2} \right)_{n=1}^\infty \notin \ell_\infty$. It follows that $Tx = (t_0, 0, 0, \dots) \in \ell_\infty$ and so $x \in \ell_\infty(T)$.

ii) If $t_n = 1$ for all $n \in \mathbb{N}$;

In this case,

$$T_n(x) = x_n - x_{n-1}$$

for all $n \in \mathbb{N}$. For $1 \leq p < \infty$, consider the sequence $x = e$ which is clearly not in ℓ_p . But, $Tx = (1, 0, 0, \dots) \in \ell_p$, that is, $x \in \ell_p(T)$.

Now, let $x = (n+1)$. Obviously, x is not in ℓ_∞ but $Tx = e \in \ell_\infty$ which means $x \in \ell_\infty(T)$.

iii) If $t_n > 1$ for all $n \in \mathbb{N}$;

Let $t_n = \frac{n+1}{n} > 1$ for all $n \geq 1$ and assume that $p = 1$. If we choose $x = \left(\frac{1}{n+1} \right) \notin \ell_1$, then $Tx = \left(\frac{1}{n(n+1)} \right) \in \ell_1$. Hence, $x \in \ell_1(T)$.

Let $t_n = \frac{n+1}{n} > 1$ for all $n \geq 1$ and choose $x = (n)$. Then, $x \notin \ell_\infty$ but $Tx = (\frac{3n+1}{n}) \in \ell_\infty$ and so $x \in \ell_\infty(T)$.

As a result, there exists $x \in \ell_p(T) \setminus \ell_p$. Thus, we conclude that the inclusion $\ell_p \subset \ell_p(T)$ strictly holds for $1 \leq p \leq \infty$. The proof is completed.

Theorem 2.10. *Neither of the spaces ℓ_∞ and $\ell_p(T)$ includes the other one, where $1 \leq p < \infty$.*

Proof. For $t_i = \sqrt{\frac{i}{i+1}}$, consider the sequence $x = \left(\prod_{i=1}^n \frac{1}{t_i^2}\right)$ which is not in ℓ_∞ . However, $x \in \ell_p(T)$ since $Tx = (t_0, 0, 0, \dots) \in \ell_p$. Now, consider the sequence $x = ((-1)^n)$ in ℓ_∞ . Thus,

$$T_n(x) = \begin{cases} t_0 & , \quad n = 0 \\ (-1)^n \left(t_n + \frac{1}{t_n}\right) & , \quad n \geq 1 \end{cases} \quad (n \in \mathbb{N}).$$

Clearly, for all $n \in \mathbb{N}$, $\left|(-1)^n \left(t_n + \frac{1}{t_n}\right)\right| > 1$ which implies that the series $\sum_n |T_n(x)|^p$ is not convergent, where $1 \leq p < \infty$. Hence, $x \notin \ell_p(T)$. We conclude that neither of the spaces includes the other.

3. The α -, β - and γ -duals of the space $\ell_p(T)$

In this section, we determine the α -, β - and γ -duals of the sequence space $\ell_p(T)$, where $1 \leq p \leq \infty$. Also, we give a **sequence** of the points of the space $\ell_p(T)$ which forms a basis for this space.

The following known results [27] and [28] are fundamental for our investigation.

Lemma 3.1. *Let $1 < p < \infty$. The following statements hold:*

(a) $A = (a_{nk}) \in (\ell_p, \ell_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk}|^q < \infty. \quad (3.1)$$

(b) $A = (a_{nk}) \in (\ell_p, c)$ if and only if (3.1) holds and

$$\lim_{n \rightarrow \infty} a_{nk} \text{ exists for all } k \in \mathbb{N}. \quad (3.2)$$

(c) $A = (a_{nk}) \in (\ell_p, c_0)$ if and only if (3.1) holds and

$$\lim_{n \rightarrow \infty} a_{nk} = 0 \text{ for all } k \in \mathbb{N}. \quad (3.3)$$

(d) $A = (a_{nk}) \in (\ell_p, \ell_1)$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_{nk} \right|^q < \infty. \quad (3.4)$$

Lemma 3.2. *The following statements hold:*

(a) $A = (a_{nk}) \in (\ell_1, \ell_\infty)$ if and only if

$$\sup_{n,k} |a_{nk}| < \infty. \quad (3.5)$$

- (b) $A = (a_{nk}) \in (\ell_1, c)$ if and only if (3.2) and (3.5)
- (c) $A = (a_{nk}) \in (\ell_1, c_0)$ if and only if (3.3) and (3.5)
- (d) $A = (a_{nk}) \in (\ell_1, \ell_1)$ if and only if

$$\sup_k \sum_n |a_{nk}| < \infty. \quad (3.6)$$

Lemma 3.3. *The following statements hold:*

- (a) $A = (a_{nk}) \in (\ell_\infty, \ell_\infty)$ if and only if (3.1) holds with $q = 1$.
- (b) $A = (a_{nk}) \in (\ell_\infty, c)$ if and only if (3.2) holds and

$$\lim_{n \rightarrow \infty} \sum_k |a_{nk}| = \sum_k \left| \lim_{n \rightarrow \infty} a_{nk} \right|. \quad (3.7)$$

- (c) $A = (a_{nk}) \in (\ell_\infty, c_0)$ if and only if (3.3) holds and

$$\lim_{n \rightarrow \infty} \sum_k |a_{nk}| = 0. \quad (3.8)$$

- (d) $A = (a_{nk}) \in (\ell_\infty, \ell_1)$ if and only if (3.4) holds with $q = 1$.

Now, we give two lemmas which are required to determine the α -, β - and γ -duals of the space $\ell_p(T)$, where $1 \leq p \leq \infty$.

Lemma 3.4. *Let $a = (a_n) \in \omega$ and the matrix $B = (b_{nk})$ be defined by $B_n = a_n T_n^{-1}$, that is,*

$$b_{nk} = \begin{cases} 0 & , \quad k > n \\ a_n g_{nk} & , \quad 0 \leq k \leq n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then, $a \in (\ell_p(T))^\alpha$ if and only if $B \in (\ell_p, \ell_1)$, where $1 \leq p \leq \infty$.

Proof. Let y be the T -transform of a sequence $x = (x_n) \in \omega$. Then, we have

$$a_n x_n = a_n T_n^{-1}(y) = B_n(y)$$

for all $n \in \mathbb{N}$. Thus, we obtain from this equality that $ax = (a_n x_n) \in \ell_1$ with $x \in \ell_p(T)$ if and only if $By \in \ell_1$ with $y \in \ell_p$. This implies that $a \in (\ell_p(T))^\alpha$ if and only if $B \in (\ell_p, \ell_1)$. The proof is completed.

Lemma 3.5. *[29, Theorem 3.1] Let $C = (c_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $U = (u_{nk})$ by*

$$c_{nk} = \begin{cases} 0 & , \quad k > n \\ \sum_{j=k}^n a_j v_{jk} & , \quad 0 \leq k \leq n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then,

$$(\ell_p(U))^\gamma = \{a = (a_k) \in \omega : C \in (\ell_p, \ell_\infty)\},$$

$$(\ell_p(U))^\beta = \{a = (a_k) \in \omega : C \in (\ell_p, c)\},$$

where $1 \leq p \leq \infty$.

Combining Lemmas 3.1-3.5 we have;

Corollary 3.6. *Let the sets $\hat{d}_1, \hat{d}_2, \hat{d}_3, \hat{d}_4, \hat{d}_5$ and $\hat{d}_6$ be defined as follows:*

$$\begin{aligned}\hat{d}_1 &= \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \left(t_k \prod_{j=k}^n \frac{1}{t_j^2} \right) a_n \right|^q < \infty \right\}, \\ \hat{d}_2 &= \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) a_j \text{ exists for each } k \in \mathbb{N} \right\}, \\ \hat{d}_3 &= \left\{ a = (a_k) \in \omega : \sup_n \sum_{k=0}^n \left| \sum_{j=k}^n \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) a_j \right|^q < \infty \right\}, \\ \hat{d}_4 &= \left\{ a = (a_k) \in \omega : \lim_{n \rightarrow \infty} \sum_{k=0}^n \left| \sum_{j=k}^n \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) a_j \right| = \sum_k \left| \sum_{j=k}^{\infty} \left(\prod_{i=k}^j \frac{t_k}{t_i^2} \right) a_j \right| < \infty \right\}, \\ \hat{d}_5 &= \left\{ a = (a_k) \in \omega : \sup_k \sum_n \left| \left(t_k \prod_{j=k}^n \frac{1}{t_j^2} \right) a_n \right| < \infty \right\}\end{aligned}$$

and

$$\hat{d}_6 = \left\{ a = (a_k) \in \omega : \sup_{n,k} \left| \sum_{j=k}^n \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) a_j \right| < \infty \right\}.$$

Then, the following statements hold:

- (a) $(\ell_p(T))^\alpha = \hat{d}_1$ and $(\ell_1(T))^\alpha = \hat{d}_5$, where $1 < p \leq \infty$.
- (b) $(\ell_p(T))^\beta = \hat{d}_2 \cap \hat{d}_3$, $(\ell_\infty(T))^\beta = \hat{d}_2 \cap \hat{d}_4$ and $(\ell_1(T))^\beta = \hat{d}_2 \cap \hat{d}_6$, where $1 < p < \infty$.
- (c) $(\ell_p(T))^\gamma = \hat{d}_3$ and $(\ell_1(T))^\gamma = \hat{d}_6$, where $1 < p \leq \infty$.

Now, we give the Schauder basis of the space $\ell_p(T)$ ($1 \leq p < \infty$).

Theorem 3.7. *Let $1 \leq p < \infty$ and define the sequence $c^{(k)} \in \ell_p(T)$ for every fixed $k \in \mathbb{N}$ by*

$$(c^{(k)})_n = \begin{cases} 0 & , \quad n < k \\ t_k \prod_{j=k}^n \frac{1}{t_j^2} & , \quad n \geq k \end{cases} \quad (n \in \mathbb{N}). \quad (3.9)$$

Then the sequence $(c^{(k)})$ is a basis for the space $\ell_p(T)$, and every $x \in \ell_p(T)$ has a unique representation of the form

$$x = \sum_k T_k(x) c^{(k)}. \quad (3.10)$$

Proof. Let $1 \leq p < \infty$. By (3.9), it is clear that $T(c^{(k)}) = e^{(k)} \in \ell_p$ and $c^{(k)} \in \ell_p(T)$ for all $k \in \mathbb{N}$.

Also, let $x \in \ell_p(T)$ given. For every non-negative integer m , we put

$$x^{(m)} = \sum_{k=0}^m T_k(x)c^{(k)}.$$

Then, we have that

$$T(x^{(m)}) = \sum_{k=0}^m T_k(x)T(c^{(k)}) = \sum_{k=0}^m T_k(x)e^{(k)}$$

and hence

$$T_n(x - x^{(m)}) = \begin{cases} 0 & (0 \leq n \leq m) \\ T_n(x) & (n > m). \end{cases}$$

Let $\epsilon > 0$ be given. Then, there exists a non-negative integer m_0 such that

$$\sum_{n=m_0+1}^{\infty} |T_n(x)|^p \leq \left(\frac{\epsilon}{2}\right)^p.$$

Therefore, we obtain for every $m \geq m_0$ that

$$\|x - x^{(m)}\|_{\ell_p(T)} = \left(\sum_{n=m+1}^{\infty} |T_n(x)|^p \right)^{1/p} \leq \left(\sum_{n=m_0+1}^{\infty} |T_n(x)|^p \right)^{1/p} \leq \frac{\epsilon}{2} < \epsilon$$

which shows that $\lim_{m \rightarrow \infty} \|x - x^{(m)}\|_{\ell_p(T)} = 0$ and hence x is represented as in (3.10).

Finally, let us show the uniqueness of the representation (3.10) of $x \in \ell_p(T)$. Assume that $x = \sum_k \mu_k(x)c^{(k)}$. Since the linear transformation T defined from $\ell_p(T)$ to ℓ_p in the proof of Theorem 2.3 is continuous, we have

$$T_n(x) = \sum_k \mu_k(x)T_n(c^{(k)}) = \sum_k \mu_k(x)\delta_{nk} = \mu_n(x) \quad (n \in \mathbb{N}).$$

Hence, the representation (3.10) of $x \in \ell_p(T)$ is unique. The proof is completed.

4. Some matrix transformations related to the sequence space $\ell_p(T)$

In the final section, we give the characterization of the classes $(\ell_p(T), \lambda)$, $(\lambda, \ell_p(T))$ and define the norm of an matrix operator in $B(\ell_p(T), \mu(S))$, where $1 \leq p \leq \infty$, $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$ and $\mu \in \{\ell_1, \ell_\infty\}$.

Throughout this section, we write $a(n, k) = \sum_{j=0}^n a_{jk}$ for given an infinite matrix $A = (a_{nk})$, where $n, k \in \mathbb{N}$.

Firstly, we give a theorem which is essential for our results.

Theorem 4.1. *Let $1 \leq p \leq \infty$ and λ be an arbitrary subset of ω . Then, we have $A = (a_{nk}) \in (\ell_p(T), \lambda)$ if and only if*

$$D^{(m)} = \left(d_{nk}^{(m)} \right) \in (\ell_p, c) \text{ for all } n \in \mathbb{N}, \quad (4.1)$$

$$D = (d_{nk}) \in (\ell_p, \lambda), \quad (4.2)$$

where $d_{nk}^{(m)} = \begin{cases} 0 & , \quad k > m \\ \sum_{j=k}^m \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) a_{nj} & , \quad 0 \leq k \leq m \end{cases}$ and $d_{nk} = \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) a_{nj}$ for all $k, m, n \in \mathbb{N}$.

Proof. To prove the theorem, we follow the similar way due to Kirişçi and Başar [6]. Let $A = (a_{nk}) \in (\ell_p(T), \lambda)$ and $x = (x_k) \in \ell_p(T)$. By (2.2), we have

$$\begin{aligned} \sum_{k=0}^m a_{nk} x_k &= \sum_{k=0}^m a_{nk} \sum_{j=0}^k \left(t_j \prod_{i=j}^k \frac{1}{t_i^2} \right) y_j \\ &= \sum_{k=0}^m \sum_{j=k}^m \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) a_{nj} y_k \\ &= \sum_{k=0}^m d_{nk}^{(m)} y_k \\ &= D_n^{(m)}(y) \end{aligned}$$

for all $m, n \in \mathbb{N}$. Since Ax exists, $D^{(m)}$ belongs to the class (ℓ_p, c) . Letting $m \rightarrow \infty$ in the last equality, we obtain $Ax = Dy$ which gives the result $D \in (\ell_p, \lambda)$.

Conversely, suppose the conditions (4.1), (4.2) hold and take any $x \in \ell_p(T)$. Then, we have $(d_{nk})_{k \in \mathbb{N}} \in \ell_p^\beta$ which gives together with (4.1) that $A_n = (a_{nk})_{k \in \mathbb{N}} \in (\ell_p(T))^\beta$ for all $n \in \mathbb{N}$. Thus, Ax exists. Therefore, we derive by the above equality as $m \rightarrow \infty$ that $Ax = Dy$, and this shows that $A \in (\ell_p(T), \lambda)$.

We obtain the following results by combining Lemma 4.1 with Lemmas 3.1-3.3.

Theorem 4.2.

(a) $A = (a_{nk}) \in (\ell_1(T), \ell_\infty)$ if and only if (3.5) holds with d_{nk} instead of a_{nk} and

$$\lim_{m \rightarrow \infty} d_{nk}^{(m)} \text{ exists } (\forall n, k \in \mathbb{N}), \quad (4.3)$$

$$\sup_{m, k} \left| d_{nk}^{(m)} \right| < \infty \quad (\forall n \in \mathbb{N}) \quad (4.4)$$

also hold.

(b) $A = (a_{nk}) \in (\ell_1(T), c)$ if and only if (4.3) and (4.4) hold, and (3.2) and (3.5) also hold with d_{nk} instead of a_{nk} .

(c) $A = (a_{nk}) \in (\ell_1(T), c_0)$ if and only if (4.3) and (4.4) hold, and (3.3) and (3.5) also hold with d_{nk} instead of a_{nk} .

(d) $A = (a_{nk}) \in (\ell_1(T), \ell_1)$ if and only if (4.3) and (4.4) hold, and (3.6) also holds with d_{nk} instead of a_{nk} .

Theorem 4.3. *Let $1 < p < \infty$.*

(a) $A = (a_{nk}) \in (\ell_p(T), \ell_\infty)$ *if and only if* (4.3) *and*

$$\sup_m \sum_{k=0}^m \left| d_{nk}^{(m)} \right|^q < \infty \quad (4.5)$$

hold, and (3.1) also hold with d_{nk} instead of a_{nk} .

(b) $A = (a_{nk}) \in (\ell_p(T), c)$ *if and only if* (4.3) *and* (4.5) *hold, and (3.1) and (3.2) also hold with d_{nk} instead of a_{nk} .*

(c) $A = (a_{nk}) \in (\ell_p(T), c_0)$ *if and only if* (4.3) *and* (4.5) *hold, and (3.1) and (3.3) also hold with d_{nk} instead of a_{nk} .*

(d) $A = (a_{nk}) \in (\ell_p(T), \ell_1)$ *if and only if* (4.3) *and* (4.5) *hold, and (3.4) also holds with d_{nk} instead of a_{nk} .*

Theorem 4.4.

(a) $A = (a_{nk}) \in (\ell_\infty(T), \ell_\infty)$ *if and only if* (4.3) *and*

$$\lim_{m \rightarrow \infty} \sum_{k=0}^m \left| d_{nk}^{(m)} \right| = \sum_k |d_{nk}| \quad \text{for each } n \in \mathbb{N} \quad (4.6)$$

hold, and (3.1) also holds with $q = 1$ and d_{nk} instead of a_{nk} .

(b) $A = (a_{nk}) \in (\ell_\infty(T), c)$ *if and only if* (4.3) *and* (4.6) *hold, and (3.2) and (3.7) also hold with d_{nk} instead of a_{nk} .*

(c) $A = (a_{nk}) \in (\ell_\infty(T), c_0)$ *if and only if* (4.3) *and* (4.6) *hold, and (3.3) and (3.8) also hold with d_{nk} instead of a_{nk} .*

(d) $A = (a_{nk}) \in (\ell_\infty(T), \ell_1)$ *if and only if* (4.3), (4.6) *and*

$$\sup_{N, K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} d_{nk} \right| < \infty$$

hold.

By using Theorems 4.2-4.4, we derive the following results:

Corollary 4.5. *The following statements hold:*

(a) $A = (a_{nk}) \in (\ell_1(T), cs_0)$ *if and only if* (3.3), (3.5) *and* (4.3), (4.4) *hold with $d(n, k)$ instead of a_{nk} and d_{nk} , respectively.*

(b) $A = (a_{nk}) \in (\ell_1(T), cs)$ *if and only if* (3.2), (3.5) *and* (4.3), (4.4) *hold with $d(n, k)$ instead of a_{nk} and d_{nk} , respectively.*

(c) $A = (a_{nk}) \in (\ell_1(T), bs)$ *if and only if* (3.5) *and* (4.3), (4.4) *hold with $d(n, k)$ instead of a_{nk} and d_{nk} , respectively.*

Corollary 4.6. *Let $1 < p < \infty$. Then, the following statements hold:*

(a) $A = (a_{nk}) \in (\ell_p(T), cs_0)$ *if and only if* (3.1), (3.3) *and* (4.3), (4.5) *hold with $d(n, k)$ instead of a_{nk} and d_{nk} , respectively.*

(b) $A = (a_{nk}) \in (\ell_p(T), cs)$ *if and only if* (3.1), (3.2) *and* (4.3), (4.5) *hold with $d(n, k)$ instead of a_{nk} and d_{nk} , respectively.*

(c) $A = (a_{nk}) \in (\ell_p(T), bs)$ *if and only if* (3.1) *and* (4.3), (4.5) *hold with $d(n, k)$ instead of a_{nk} and d_{nk} , respectively.*

Corollary 4.7. *The following statements hold:*

- (a) $A = (a_{nk}) \in (\ell_\infty(T), cs_0)$ if and only if (3.3), (3.8) and (4.3), (4.6) hold with $d(n, k)$ instead of a_{nk} and d_{nk} , respectively.
- (b) $A = (a_{nk}) \in (\ell_\infty(T), cs)$ if and only if (3.2), (3.7) and (4.3), (4.6) hold with $d(n, k)$ instead of a_{nk} and d_{nk} , respectively.
- (c) $A = (a_{nk}) \in (\ell_\infty(T), bs)$ if and only if (3.1) with $q = 1$ and (4.3), (4.6) hold with $d(n, k)$ instead of a_{nk} and d_{nk} , respectively.

Now, we introduce the matrix transformations from the space $\lambda \in \{\ell_1, c_0, c, \ell_\infty\}$ to $\ell_p(T)$, where $1 \leq p \leq \infty$. Before this, we give the necessary and sufficient conditions for the matrix transformation A is in (λ, ℓ_p) .

Lemma 4.8. *The following statements hold:*

- (a) $A \in (\ell_\infty, \ell_p) = (c, \ell_p) = (c_0, \ell_p)$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_{nk} \right|^p < \infty, \text{ where } 1 \leq p < \infty. \quad (4.7)$$

- (b) $A \in (\ell_\infty, \ell_\infty) = (c, \ell_\infty) = (c_0, \ell_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk}| < \infty. \quad (4.8)$$

- (c) $A \in (\ell_1, \ell_p)$ if and only if

$$\sup_k \sum_n |a_{nk}|^p < \infty, \text{ where } 1 \leq p < \infty. \quad (4.9)$$

When we change the roles of the spaces $\ell_p(T)$ and ℓ_p with λ in Theorem 4.1, we obtain the following theorem.

Theorem 4.9. *Assume that there exists the following relation between the terms of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$*

$$b_{nk} = -\frac{1}{t_n} a_{n-1,k} + t_n a_{nk} \quad (4.10)$$

for all $k, n \in \mathbb{N}$ and λ be any given sequence space. Then, $A \in (\lambda, \ell_p(T))$ if and only if $B \in (\lambda, \ell_p)$, where $1 \leq p \leq \infty$.

Proof. Let $x = (x_k) \in \lambda$. Then, by using the relation (4.10) one can easily obtain the following equality

$$\sum_{k=0}^m b_{nk} x_k = \sum_{k=0}^m \left(-\frac{1}{t_n} a_{n-1,k} + t_n a_{nk} \right) x_k \text{ for all } m, n \in \mathbb{N}$$

which yields as $m \rightarrow \infty$ that $(B_n(x)) = (T_n(Ax))$. Therefore, we conclude that $Ax \in \ell_p(T)$ for $x \in \lambda$ if and only if $Bx \in \ell_p$ for $x \in \lambda$, where $1 \leq p \leq \infty$. The proof is completed.

By combining Lemma 3.2 (a), Lemma 4.8 and Theorem 4.9, we obtain the following results:

Corollary 4.10. *Let the matrices $A = (a_{nk})$ and $B = (b_{nk})$ be connected by (4.10). Then, we obtain:*

(a) $A = (a_{nk}) \in (\ell_\infty, \ell_1(T)) = (c, \ell_1(T)) = (c_0, \ell_1(T))$ if and only if (4.7) holds with $p = 1$ and b_{nk} instead of a_{nk} .

(b) $A = (a_{nk}) \in (\ell_1, \ell_1(T))$ if and only if (4.9) holds with $p = 1$ and b_{nk} instead of a_{nk} .

Corollary 4.11. *Let the matrices $A = (a_{nk})$ and $B = (b_{nk})$ be connected by (4.10). For $1 < p < \infty$, we obtain:*

(a) $A = (a_{nk}) \in (\ell_\infty, \ell_p(T)) = (c, \ell_p(T)) = (c_0, \ell_p(T))$ if and only if (4.7) holds with b_{nk} instead of a_{nk} .

(b) $A = (a_{nk}) \in (\ell_1, \ell_p(T))$ if and only if (4.9) holds with b_{nk} instead of a_{nk} .

Corollary 4.12. *Let the matrices $A = (a_{nk})$ and $B = (b_{nk})$ be connected by (4.10). Then, we obtain:*

(a) $A = (a_{nk}) \in (\ell_\infty, \ell_\infty(T)) = (c, \ell_\infty(T)) = (c_0, \ell_\infty(T))$ if and only if (4.8) holds with b_{nk} instead of a_{nk} .

(b) $A = (a_{nk}) \in (\ell_1, \ell_\infty(T))$ if and only if (3.5) holds with b_{nk} instead of a_{nk} .

Now, we investigate the norm of the infinite matrices in the class $B(\ell_p(T), \mu(S))$, where $\mu \in \{\ell_1, \ell_\infty\}$ and $1 \leq p \leq \infty$. Firstly, we give an essential lemma for our investigation.

Lemma 4.13. *Let $B = (b_{nk})$ be an infinite matrix. Then the following statements hold:*

(a) *The norm of B in $B(\ell_p, \ell_\infty)$ is defined by*

$$\|B\|_{(\ell_1, \ell_\infty)} = \sup_{n,k} |b_{nk}|$$

and

$$\|B\|_{(\ell_p, \ell_\infty)} = \sup_n \sum_k |b_{nk}|^q \quad (1 < p \leq \infty).$$

(b) *The norm of B in $B(\ell_p, \ell_1)$ is defined by*

$$\|B\|_{(\ell_1, \ell_1)} = \sup_k \sum_n |b_{nk}|$$

and

$$\|B\|_{(\ell_p, \ell_1)} = \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} b_{nk} \right|^q \quad (1 < p \leq \infty).$$

Theorem 4.14. *Let T and S be two band matrices given by the sequences $t = (t_n)$ and $s = (s_n)$, respectively, and $A = (a_{nk})$ be an infinite matrix.*

(a) *If $A \in B((\ell_1(T), \ell_\infty(S)))$, then*

$$M_1 = \sup_{n,k} \left| \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) \left(s_n a_{nj} - \frac{1}{s_n} a_{n-1,j} \right) \right|$$

is finite. In this case, $\|A\|_{(\ell_1(T), \ell_\infty(S))} = M_1$.

(b) Let $1 < p \leq \infty$. If $A \in B((\ell_p(T), \ell_\infty(S)))$, then

$$M_p = \sup_n \sum_k \left| \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) \left(s_n a_{nj} - \frac{1}{s_n} a_{n-1,j} \right) \right|^q$$

is finite. In this case, $\|A\|_{(\ell_p(T), \ell_\infty(S))} = M_p$.

(c) If $A \in B((\ell_1(T), \ell_1(S)))$, then

$$K_1 = \sup_k \sum_n \left| \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) \left(s_n a_{nj} - \frac{1}{s_n} a_{n-1,j} \right) \right|$$

is finite. In this case, $\|A\|_{(\ell_1(T), \ell_1(S))} = K_1$.

(d) Let $1 < p \leq \infty$. If $A \in B((\ell_p(T), \ell_1(S)))$, then

$$K_p = \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \sum_{j=k}^{\infty} \left(t_k \prod_{i=k}^j \frac{1}{t_i^2} \right) \left(s_n a_{nj} - \frac{1}{s_n} a_{n-1,j} \right) \right|^q$$

is finite. In this case, $\|A\|_{(\ell_p(T), \ell_1(S))} = K_p$.

Proof. From Theorem 2.3, we know that T is an isometric isomorphism: $\ell_p(T) \rightarrow \ell_p$, where $1 \leq p \leq \infty$. Let $B = SAT^{-1}$. Thus, the following diagram

$$\begin{array}{ccc} \ell_p(T) & \xrightarrow{A} & \mu(S) \\ T^{-1} \uparrow & & \downarrow S \\ \ell_p & \xrightarrow[B=SAT^{-1}]{} & \mu \end{array}$$

shows that $\|A\|_{(\ell_p(T), \mu(S))} = \|B\|_{(\ell_p, \mu)}$, where $\mu \in \{\ell_\infty, \ell_1\}$ and $1 \leq p \leq \infty$. Using Lemma 4.13 (a) and (b), we have that

$$\|B\|_{(\ell_p, \mu)} = \begin{cases} M_p, & \text{if } \mu = \ell_\infty \\ K_p, & \text{if } \mu = \ell_1, \end{cases}$$

where $1 \leq p \leq \infty$.

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