

V_4 - Magic Labelings of Some Wheel Related Graphs

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Abstract

The Klein 4-group, denoted by V_4 is an abelian group of order 4. It has elements $V_4 = \{0, a, b, c\}$, with $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$. A graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$, is said to be V_4 magic if there exists a labeling $\ell : E(G) \rightarrow V_4 \setminus \{0\}$ such that the induced vertex labeling $\ell^+ : V(G) \rightarrow V_4$ defined by

$$\ell^+(u) = \sum \{\ell(uv) : uv \in E(G)\}$$

is a constant map. If this constant is equal to a , we say that ℓ is an a -sum V_4 magic labeling of G . Any graph that admits an a -sum V_4 magic labeling is called an a -magic V_4 graph. When this constant is 0 we call G a zero-sum V_4 -magic graph. We divide the class of V_4 magic graphs into the following three categories:

- (i) $\mathcal{V}_a$, the class of a -sum V_4 magic graphs,
- (ii) $\mathcal{V}_0$, the class of zero-sum V_4 magic graphs,
- (iii) $\mathcal{V}_{a,0}$, the class of graphs which are both a -sum and zero-sum V_4 magic.

In this paper, we identify some cycle related graphs which belong to the above categories.

Keywords: Klein 4-group; V_4 - magic graph; Wheel graph

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1 Introduction

In this paper all graphs are connected, finite, simple, and undirected. For graph theory notations and terminology not directly defined in this paper, we refer readers to [1]. For an abelian group A , written additively, any mapping $\ell : E(G) \rightarrow A \setminus \{0\}$ is called a labeling, where 0 denote the identity element in A .

For any abelian group A , a graph $G = (V, E)$ is said to be A -magic if there exists a labeling $\ell : E(G) \rightarrow A \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \rightarrow A$ defined by

$$\ell^+(v) = \sum \{\ell(uv) : uv \in E(G)\}$$

is a constant map [2]. If $\ell : E(G) \rightarrow A \setminus \{0\}$ ($|A| > 2$) is a magic labeling of G with sum c , then $-\ell : E(G) \rightarrow A \setminus \{0\}$, defined by $(-\ell)(u) = -\ell(u)$ is another A -magic labeling of G with sum $-c$. The labeling $-\ell$ is called the inverse of ℓ . This implies that A -magic labeling of a graph need not be unique. A graph $G = (V, E)$ is called non-magic if for every abelian group A , the graph is not A -magic [2]. The most obvious example of a non-magic graph is P_n ($n > 3$), the path of order n . As a result, any graph with a path pendant of length at least two would be non-magic.

The Klein 4-group, denoted by V_4 is an abelian group of order 4. It has elements $V_4 = \{0, a, b, c\}$, with $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$. Note that V_4 is not cyclic, since every element has order 2 (except for the identity, of course) and V_4 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The V_4 magic graphs was introduced by S. M. Lee et al. in 2002 [2]. There has been increased interest in the study of V_4 magic graphs since the publication of [2]. We define, a graph G is an a -sum V_4 magic if there exists a labeling $\ell : E(G) \rightarrow V_4 \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \rightarrow A$ satisfies $\ell^+(v) = a$ for all $v \in V(G)$ ($a \neq 0$). If $\ell^+(v) = 0$, for all $v \in V(G)$, we say that the graph is zero-sum V_4 magic. We divide the class of V_4 magic graphs into the following three categories:

- (i) $\mathcal{V}_a$, the class of a -sum V_4 magic graphs,
- (ii) $\mathcal{V}_0$, the class of zero-sum V_4 magic graphs,
- (iii) $\mathcal{V}_{a,0}$, the class of graphs which are both a -sum and zero-sum V_4 magic.

Note that $K_3 \in \mathcal{V}_0$, but $K_3 \notin \mathcal{V}_a$ and, $C_4 \in \mathcal{V}_{a,0}$. In this paper, we identify some cycle related graphs which belong to the above three categories.

2 Cycle Related Graphs

Definition 2.1. A path is an ordered list of distinct vertices $u_1, u_2, \dots, u_n$ such that $u_{i-1}u_i$ is an edge for all $i, 2 \leq i \leq n$. The ordered list of vertices $u_1, u_2, \dots, u_n$ is a cycle if u_nu_1 is also an edge. Paths and cycles of n vertices are often denoted by P_n and C_n , respectively. Throughout this paper the suffices of the vertices u_i in the cycle C_n are taken modulo n .

We start with the following lemma.

Lemma 2.1. *If $\ell : E(C_n) \rightarrow V_4 \setminus \{0\}$ is a labeling of C_n , then*

$$\sum_{i=1}^n \ell^+(u_i) = 0,$$

where $u_1, u_2, \dots, u_n$ are the vertices of C_n .

Proof. Let ℓ be a labeling of C_n . Then we have, $\ell^+(u_i) = \ell(u_{i-1}u_i) + \ell(u_iu_{i+1})$. This implies that $\sum_{i=1}^n \ell^+(u_i) = 0$. This completes the proof. $\square$

Using lemma 2.1, we prove the following result:

Theorem 2.1. $C_n \in \mathcal{V}_a$ if and only if n is even.

Proof. Assume that $C_n \in \mathcal{V}_a$. Then by lemma 2.1, we have $na = 0$. This implies that n is even. Conversely, assume that n is even. Define $\ell : E(C_n) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}\ell(u_i u_{i+1}) &= b \quad \text{for } i = 1, 3, \dots, n-1, \\ \ell(u_i u_{i+1}) &= c \quad \text{for } i = 2, 4, \dots, n.\end{aligned}$$

Obviously, $\ell^+(u_i) = b + c = a$, for $i = 1, 2, \dots, n$. Thus ℓ is an a -sum V_4 magic labeling of C_n . This completes the proof. $\square$

Theorem 2.2. $C_n \in \mathcal{V}_0$ for all $n \geq 3$.

Proof. If we label all the edges of C_n by a , then we obtain $\ell^+(u_i) = 0$ for $i = 1, 2, \dots, n$. $\square$

Theorem 2.3. If n is even, then $C_n \in \mathcal{V}_{a,0}$.

Proof. Proof follows from theorems 2.1 and 2.2. $\square$

Definition 2.2. We denote by $C(n, \underbrace{k_1, k_2, \dots, k_t}_t)$ the set of all graphs obtained by identifying the apex vertices of t stars K_{1,k_i} ($i = 1, 2, \dots, t$) with t ($1 \leq t \leq n$) vertices of C_n . Observe that $C(n, \underbrace{k, k, \dots, k}_n)$ is a unique graph. Four members of $C(16, 7, 7, 7, 7)$ are shown in figures 1 and 2.

Also two members of $C(15, 7, 7, 7)$ are shown in figure 3.

Theorem 2.4. If $C(n, k_1, k_2, \dots, k_t) \subset \mathcal{V}_a$, then $n + k_1 + k_2 + \dots + k_t$ is even.

Proof. Observe that each member of $C(n, k_1, k_2, \dots, k_t)$ has $n + k_1 + k_2 + \dots + k_t$ vertices. One can verify that $(n + k_1 + k_2 + \dots + k_t)a = 0$. This implies that $n + k_1 + k_2 + \dots + k_t$ is even. $\square$

Conjecture 2.5. If $n + k_1 + k_2 + \dots + k_t$ is even, then $C(n, k_1, k_2, \dots, k_t) \subset \mathcal{V}_a$.

We prove some special cases of conjecture 2.5.

Theorem 2.6. If $n + tk$ is even, then $C(n, \underbrace{k, k, \dots, k}_t) \subset \mathcal{V}_a$.

Proof. Consider a graph G in the set $C(n, \underbrace{k, k, \dots, k}_t)$. We consider 4 cases:

Case 1: Suppose n, k and t are even. In this case, we label all edges of C_n as described in the proof of theorem 2.1 and label all pendant edges by a . Then obviously this is an a -sum V_4 magic labeling of G .

Case 2: Suppose n, k are even and t is odd. In this case, the labeling is exactly similar to case 1.

Case 3: Suppose n and t are even and k is odd. Without loss of generality assume that apex vertices of the t stars are at $u_1, u_{i_1}, u_{i_2}, \dots, u_{i_t}$, $1 < i_1 < i_2 < \dots < i_t$ of the cycle C_n . First, label all pendant edges by a . We label the edges of C_n as follows:

Consider the vertex u_{i_1} . If i_1 is even, we label the edges $u_n u_1, u_1 u_2, \dots, u_{i_1} u_{i_1+1}$ as follows

$$\begin{aligned}\ell(u_n u_1) &= b, \\ \ell(u_i u_{i+1}) &= b, \quad \text{for } i = 1, 3, \dots, i_1 - 1 \\ \ell(u_i u_{i+1}) &= c, \quad \text{for } i = 2, 4, \dots, i_1 - 2 \\ \ell(u_{i_1}, u_{i_1+1}) &= b.\end{aligned}$$

If i_1 is odd, we label the edges $u_n u_1, u_1 u_2, \dots, u_{i_1} u_{i_1+1}$ as follows

$$\begin{aligned}\ell(u_n u_1) &= b, \\ \ell(u_i u_{i+1}) &= b, \text{ for } i = 1, 3, \dots, i_1 - 2 \\ \ell(u_i u_{i+1}) &= c, \text{ for } i = 2, 4, \dots, i_1 - 1 \\ \ell(u_{i_1}, u_{i_1+1}) &= c.\end{aligned}$$

So, we have

$$\ell(u_{i_1-1} u_{i_1}) = \ell(u_{i_1} u_{i_1+1}) = \begin{cases} b & \text{if } i_1 \text{ even} \\ c & \text{if } i_1 \text{ odd} \end{cases}$$

Therefore,

$$\ell^+(u_{i_1}) = \begin{cases} b + b + ta = a & \text{if } i_1 \text{ even} \\ c + c + ta = a & \text{if } i_1 \text{ odd,} \end{cases}$$

Furthermore,

$$\ell^+(u_i) = \begin{cases} b + b + ka = a & \text{if } i = 1, \\ b + c = a & \text{if } i = 2, 3, \dots, i_1 - 1, \\ b + b + ka = a & \text{if } i = i_1 \text{ and } i_1 \text{ is even,} \\ c + c + ka = a & \text{if } i = i_1 \text{ and } i_1 \text{ is odd.} \end{cases}$$

Next, consider the vertex u_{i_2} . Here we consider the following cases:

i_1 and i_2 are even: In this case we label the edges $u_{i_1+1} u_{i_1+2}, u_{i_1+2} u_{i_1+3}, \dots, u_{i_2} u_{i_2+1}$ consecutively by $c, b, c, b, \dots, c, b, c, c$.

i_1 is even and i_2 is odd : In this case we label the edges $u_{i_1+1} u_{i_1+2}, u_{i_1+2} u_{i_1+3}, \dots, u_{i_2} u_{i_2+1}$ consecutively by $c, b, c, b, \dots, c, b, b$.

i_1 is odd and i_2 is even : In this case we label the edges $u_{i_1+1} u_{i_1+2}, u_{i_1+2} u_{i_1+3}, \dots, u_{i_2} u_{i_2+1}$ consecutively by $b, c, b, c, \dots, b, c, c$.

i_1 and i_2 are odd : In this case we label the edges $u_{i_1+1} u_{i_1+2}, u_{i_1+2} u_{i_1+3}, \dots, u_{i_2} u_{i_2+1}$ consecutively by $b, c, b, c, \dots, c, b, b$.

Proceeding like this, we eventually arrive at u_{i_t} . If i_t is even, then obviously $\ell(u_{i_t} u_{i_t-1}) = \ell(u_{i_t} u_{i_t+1}) = c$. Then label the edges $u_{i_t+1} u_{i_t+2}, u_{i_t+2} u_{i_t+3}, \dots, u_{n-2} u_{n-1}$ consecutively by $b, c, b, c, \dots, b, c$. If i_t is odd, then obviously $\ell(u_{i_t} u_{i_t-1}) = \ell(u_{i_t} u_{i_t+1}) = b$. Then label the edges $u_{i_t+1} u_{i_t+2}, u_{i_t+2} u_{i_t+3}, \dots, u_{n-2} u_{n-1}$ consecutively by $c, b, c, b, \dots, c$. Obviously this labeling is an a -sum v_4 magic labeling of G .

case 4: n, k and t are odd. In this case, the labeling is similar to case 3. Some members of $C(16, 7, 7, 7, 7)$, $C(15, 7, 7, 7)$ and its labelings are shown in figures 1,2 and 3.

This completes the proof. $\square$

Definition 2.3. The corona $G_1 \odot G_2$ of graphs G_1 and G_2 is the graph obtained by taking one copy of G_1 , which has p_1 vertices, and p_1 copies of G_2 , and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 [3].

Definition 2.4. The sun on $2n$ vertices is a corona of the form $C_n \odot K_1$ where $n \geq 3$. The sun $C_n \odot K_1$ is denoted by Sun_n . A broken sun is a connected unicyclic subgraph of a sun. We denote by $BS(p, q)$ the set of broken suns with $n = p + q$ vertices and with a p -cycle, note that $BS(p; p) = C_p \odot K_1$. For $p > 2$ and $0 < q < p$, a consecutive broken sun, denoted by $CBSun_{p,q}$ is the graph belonging to $BS(p, q)$ such that the subgraph induced by the vertices of degree 2 is a path on $p - q$ vertices. A broken sun (or a sun) is odd (resp. even) if p is odd (resp. even)[3]. Some examples are depicted in Figure 4.

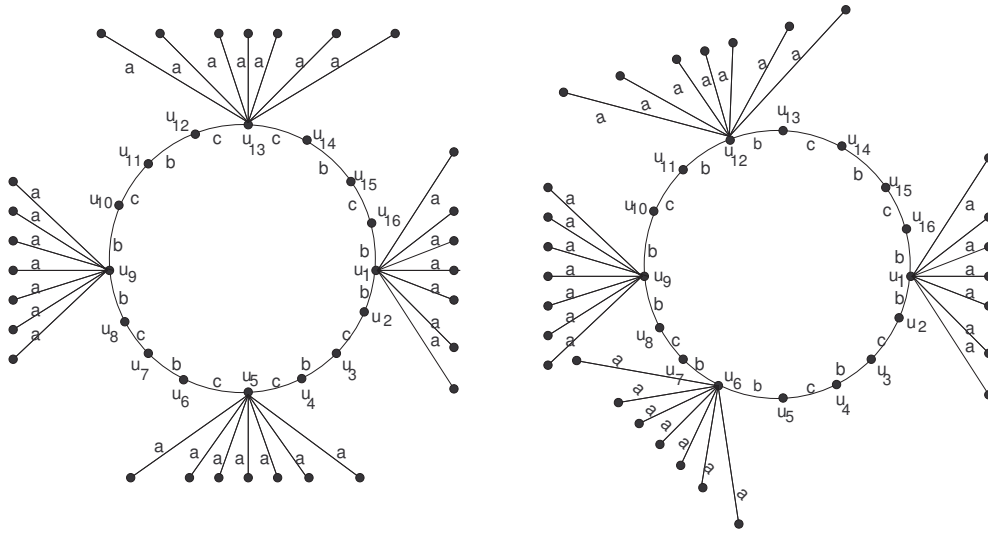


Figure 1: Figure showing members of $C(16, 7, 7, 7, 7)$ and its labelings

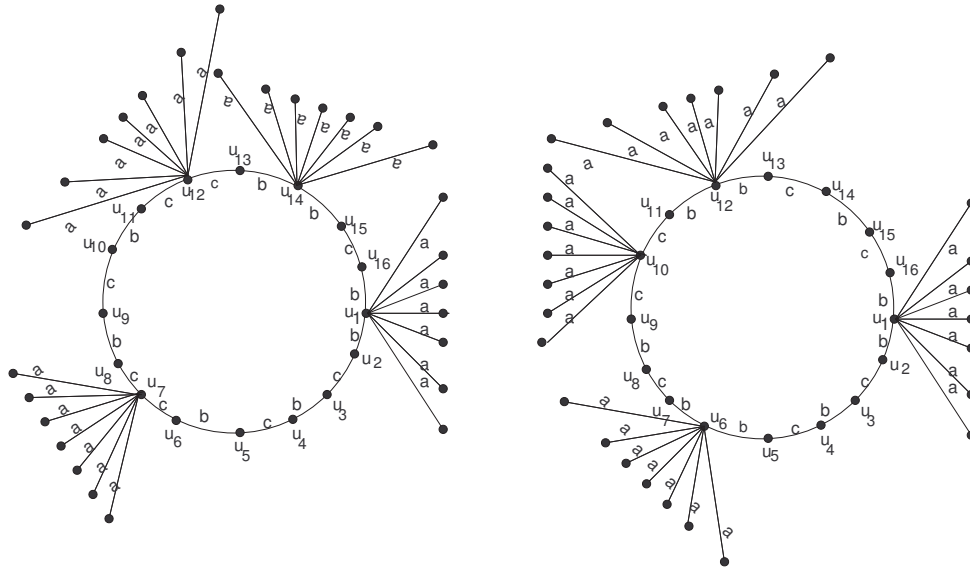


Figure 2: Figure showing members of $C(16, 7, 7, 7, 7)$ and its labelings

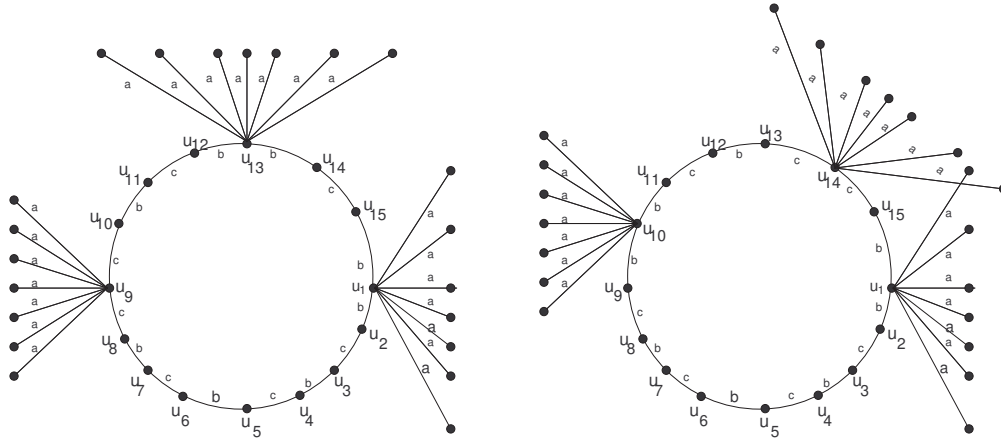


Figure 3: Figure showing some members of $C(15, 7, 7, 7)$ and its labelings

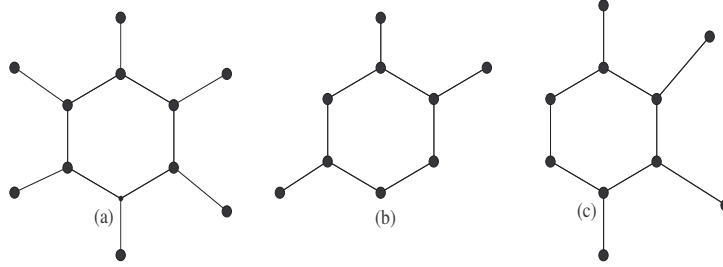


Figure 4: (a) Sun_6 , (b) $BSun(6, 3)$, (c) $CBSun(6, 4)$

Theorem 2.7. $Sun_n \in \mathcal{V}_a$ for all n .

Proof. Observe that $Sun_n = C(n, \underbrace{1, 1, \dots, 1}_n)$. So, the proof of the theorem follows from theorem 2.6. □

Theorem 2.8. $Sun_n \notin \mathcal{V}_0$ for all n .

Proof. Since Sun_n has pendant edges, $Sun_n \notin \mathcal{V}_0$. □

Theorem 2.9. $BSun(p, q) \subset \mathcal{V}_a$ if and only if $p + q$ is even.

Proof. Observe that any member in $BSun(p, q)$ has $p + q$ number of vertices. If $BSun(p, q) \in \mathcal{V}_a$, then $(p + q)a = 0$. This implies that $p + q$ is even. Converse part follows from theorem 2.6. □

Theorem 2.10. $CBSun(p, q) \subset \mathcal{V}_a$ if and only if $p + q$ is even.

Proof. Proof follows from theorem 2.6. □

Theorem 2.11. $C_n \odot K_2 \in \mathcal{V}_a$ if and only if n is even.

Proof. Note that $C_n \odot K_2$ has $3n$ vertices. If $C_n \odot K_2 \in \mathcal{V}_a$, then $3na = 0$. This implies that $na = 0$. Consequently, n is even. Converse part is trivial. □

Theorem 2.12. $C_n \odot K_2 \in \mathcal{V}_0$ for all $n \geq 3$.

Proof. Obvious. □

Theorem 2.13. If n is even, then $C_n \odot K_2 \in \mathcal{V}_{a,0}$.

Proof. Proof follows from theorems 2.11 and 2.12. □

Theorem 2.14. $C_n \odot C_m \in \mathcal{V}_a$ if and only if $n(m+1)$ is even.

Proof. Suppose $C_n \odot C_m \in \mathcal{V}_a$. Then, $n(m+1)a = 0$. This implies that $n(m+1)$ is even.

Conversely assume that $n(m+1)$ is even. Let the vertices of C_n be $(u_1, u_2, \dots, u_n, u_1)$ and let the vertices of C_m be $v_1, v_2, \dots, v_m$. We consider the following cases

Case 1: Assume that n is even and m is odd. Define $\ell : E(C_n \odot C_m) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_i u_{i+1}) &= a \quad \text{for } i = 1, 2, \dots, n, \\ \ell(v_i v_{i+1}) &= a \quad \text{for } i = 1, 2, \dots, m. \\ \text{for } j = 1, 2, \dots, n : \\ \ell(u_j v_i) &= a \quad \text{for } i = 1, 2, \dots, m. \\ \text{end for} \end{aligned}$$

Obviously ℓ is an a -sum V_4 magic labeling of $C_n \odot C_m$.

Case 2: Suppose m and n are odd. In this case the labeling is exactly similar to case 1.

Case 3: Suppose both m and n are even. Define $\ell : E(C_n \odot C_m) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_i u_{i+1}) &= b \quad \text{for } i = 1, 3, \dots, n-1, \\ \ell(u_i u_{i+1}) &= c \quad \text{for } i = 2, 4, \dots, n, \\ \ell(v_i v_{i+1}) &= a \quad \text{for } i = 1, 2, \dots, m. \\ \text{for } j = 1, 2, \dots, n : \\ \ell(u_j v_i) &= a \quad \text{for } i = 1, 2, \dots, m. \\ \text{end for} \end{aligned}$$

This completes the proof. The graph $C_{12} \odot C_5$ is shown in figure 5. □

Theorem 2.15. $C_n \odot C_m \in \mathcal{V}_0$ for all $m \geq 3$ and $n \geq 3$.

Proof. Let $V(C_n) = \{u_1, u_2, \dots, u_n\}$ and $V(C_m) = \{v_1, v_2, \dots, v_m\}$. We consider the following cases:

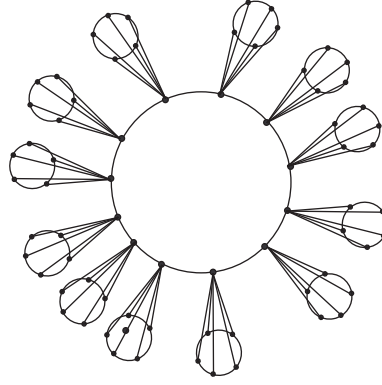


Figure 5: The graph: $C_{12} \odot C_5$

Case 1: Suppose n and m are even. Define $\ell : E(C_n \odot C_m) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_i u_{i+1}) &= a && \text{for } i = 1, 2, \dots, n, \\ \ell(v_i v_{i+1}) &= b && \text{for } i = 1, 3, \dots, m-1, \\ \ell(v_i v_{i+1}) &= c && \text{for } i = 2, 4, \dots, m, \\ \text{for } j = 1, 2, \dots, n : \\ &\ell(u_j v_i) = a && \text{for } i = 1, 2, \dots, m. \\ \text{end for} \end{aligned}$$

Obviously ℓ is a zero sum V_4 magic labeling of $C_n \odot C_m$.

Case 2: Suppose n is even and m is odd: Define $\ell : E(C_n \odot C_m) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_i u_{i+1}) &= a && \text{for } i = 1, 2, \dots, n, \\ \ell(v_i v_{i+1}) &= b && \text{for } i = 1, 3, \dots, m-2, \\ \ell(v_i v_{i+1}) &= c && \text{for } i = 2, 4, \dots, m-1, \\ \ell(v_m v_1) &= a, \\ \text{for } j = 1, 2, \dots, n : \\ &\ell(u_j v_i) = a && \text{for } i = 1, 2, \dots, m-1, \\ &\ell(u_j v_m) = b, \\ &\ell(u_j v_1) = c. \\ \text{end for} \end{aligned}$$

Then we have

$$\begin{aligned} \ell(u_i) &= a + a + (m-2)a + b + c = 0, && \text{for } i = 1, 2, \dots, n, \\ \ell(v_i) &= a + b + c = 0, && \text{for } i = 1, 2, \dots, m. \end{aligned}$$

Case 3: Suppose m and n are odd. In this case the labeling is exactly similar to case 2.

Case 4: Suppose n is odd and m is even. In this case label all edges of $C_n \odot C_m$ by a .

This completes the proof. $\square$

Theorem 2.16. *If $n(m+1)$ is even, then $C_n \odot K_m \in \mathcal{V}_{a,0}$.*

Proof. Proof follows from theorems 2.14 and 2.15. $\square$

Theorem 2.17. *$C_n \odot K_m \in \mathcal{V}_a$ if and only if $n(m+1)$ is even.*

Proof. Observe that $C_n \odot K_m$ has $n + mn$ vertices. If $C_n \odot K_m \in \mathcal{V}_a$, then we have $(m+1)na = 0$. This implies that $n(m+1)$ is even. We consider 3 cases:

Let the vertices of C_n be $u_1, u_2, \dots, u_n$. We denote the j^{th} copy of K_m by K_m^j . Let the vertices of K_m^j be $\{v_{j,1}, v_{j,2}, \dots, v_{j,m}\}$.

Case 1: Suppose n is even and m is odd. In this case, first we label all edges of K_m^j by a , $j = 1, 2, \dots, n$. Next, label all edges of C_n by a . Finally, label all edges $u_i u_{j,r}$ by a for $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$; $r = 1, 2, \dots, m$. Obviously, this is an a -sum V_4 magic labeling of $C_n \odot K_m$.

Case 2: Suppose n is even and m is even. In this case, first we label all edges of K_m^j by b , $j = 1, 2, \dots, n$. Next, label all edges of C_n by $b, c, b, c, \dots$ consecutively. Finally, label all edges $u_i u_{j,r}$ by b for $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$; $r = 1, 2, \dots, m$. Obviously, this is an a -sum V_4 magic labeling of $C_n \odot K_m$.

Case 3: Suppose n and m are odd. In this case, first we label all edges of K_m^j by b , $j = 1, 2, \dots, n$. Next, label all edges of C_n by a . Finally, label all edges $u_i v_{j,r}$ by a for $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$; $r = 1, 2, \dots, m$. Obviously, this is an a -sum V_4 magic labeling of $C_n \odot K_m$.

This completes the proof. $\square$

Theorem 2.18. *$C_n \odot \overline{K_m} \in \mathcal{V}_a$ if and only if $n(m+1)$ is even, where $\overline{K_m}$ is the complement of the complete graph with m vertices.*

Proof. Note that the graph $C_n \odot \overline{K_m}$ has $n(m+1)$ vertices. If $C_n \odot \overline{K_m} \in \mathcal{V}_a$, then we have $n(m+1)$ is even. Conversely, assume that $n(m+1)$ is even. Consider n copies of $\overline{K_m}$. Let $\overline{K_m}^j$ denotes the j^{th} copy of $\overline{K_m}$. Let

$$\begin{aligned} V(C_n) &= \{u_1, u_2, \dots, u_n\}, \\ V(\overline{K_m}^j) &= \{v_{j,1}, v_{j,2}, \dots, v_{j,m}\}, \quad j = 1, 2, \dots, n. \end{aligned}$$

We consider 3 cases:

Case 1: Suppose n is even and m is odd. Define $\ell : V(C_n \odot \overline{K_m}) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} &\text{for } i = 1, 2, \dots, n : \\ &\quad \ell(u_i v_{j,r}) = a, \quad j = 1, 2, \dots, n; r = 1, 2, \dots, m \\ &\quad \ell(u_i u_{i+1}) = a \\ &\text{end for} \end{aligned}$$

Then, we have

$$\begin{aligned} \ell^+(u_i) &= a + a + ma = a \\ \ell^+(v_{j,r}) &= a, \quad j = 1, 2, \dots, n; r = 1, 2, \dots, m. \end{aligned}$$

Case 2: Suppose n and m are even. Define $\ell : V(C_n \odot \overline{K_m}) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_i u_{i+1}) &= b, \text{ for } i = 1, 3, \dots, n-1, \\ \ell(u_i u_{i+1}) &= c, \text{ for } i = 2, 4, \dots, n \\ \text{for } i &= 1, 2, \dots, n : \\ \ell(u_i v_{j,r}) &= a, \text{ } j = 1, 2, \dots, n; r = 1, 2, \dots, m \\ \text{end for} \end{aligned}$$

Then, we have

$$\begin{aligned} \ell^+(u_i) &= b + c + ma = a \\ \ell^+(v_{j,r}) &= a, \text{ } j = 1, 2, \dots, n; r = 1, 2, \dots, m. \end{aligned}$$

Case 3: Suppose n and m are odd. In this case, the labeling is exactly similar to case 1.

This completes the proof. □

Theorem 2.19. $C_n \odot \overline{K_m} \notin \mathcal{V}_0$ for all m and n .

Proof. Obvious. □

Theorem 2.20. If $n(m+1)$ is even, then $C_n \odot \overline{K_m} \in \mathcal{V}_{a,0}$.

Proof. Proof follows from 2.18 and 2.19. □

A graph G with a fixed vertex $u \in V(G)$ will be denoted by the ordered pair (G, u) . Given two ordered pairs (G, u) and (H, v) , one can construct another graph by linking these two graphs through identifying the vertices u and v . We will use the notation $(G, u) \diamond (H, v)$ for this construction or simply $G \diamond H$ if there is no ambiguity regarding the choices of u and v [4].

Definition 2.5. Given n graphs $G_i (i = 1, 2, \dots, n)$, the chain $G_1 \diamond G_2 \diamond \dots \diamond G_n$ is the graph in which one of the vertices of G_i is identified with one of the vertices of G_{i+1} . If $G_i = G$, we use the notation $\diamond G_n$ for the n -link chain all of whose links are G [4].

Theorem 2.21. $C_m \diamond C_n \in \mathcal{V}_a$ if and only if $m+n$ is odd.

Proof. Let the vertices of C_m and C_n be respectively, $u_1, u_2, \dots, u_m$ and $v_1, v_2, \dots, v_n$. Assume that u_1 and v_1 are identified with a new vertex w . Then we have, $\sum_{i=2}^m \ell^+(u_i) + \sum_{i=2}^n \ell^+(v_i) + \ell^+(w) = 0$. This implies that $(m+n)$ is odd.

Conversely, assume that $m+n$ is odd. Then we consider two cases:

Case 1: Suppose m is even and n is odd. Define a mapping $\ell : E(C_m \diamond C_n) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_i u_{i+1}) &= \begin{cases} b & \text{for } i = 1, 3, \dots, m-1, \\ c & \text{for } i = 2, 4, \dots, m, \end{cases} \\ \ell(v_i v_{i+1}) &= \begin{cases} c & \text{for } i = 1, 3, \dots, n, \\ b & \text{for } i = 2, 4, \dots, n-1. \end{cases} \end{aligned}$$

Clearly ℓ is an a -sum magic labeling of $C_m \diamond C_n$.

Case 2: Suppose m is odd and n is even. The remaining part is exactly similar to case 1.

This completes the proof. □

Theorem 2.22. $\diamond[C_n]_m \in \mathcal{V}_a$ if and only if m is odd and n is even.

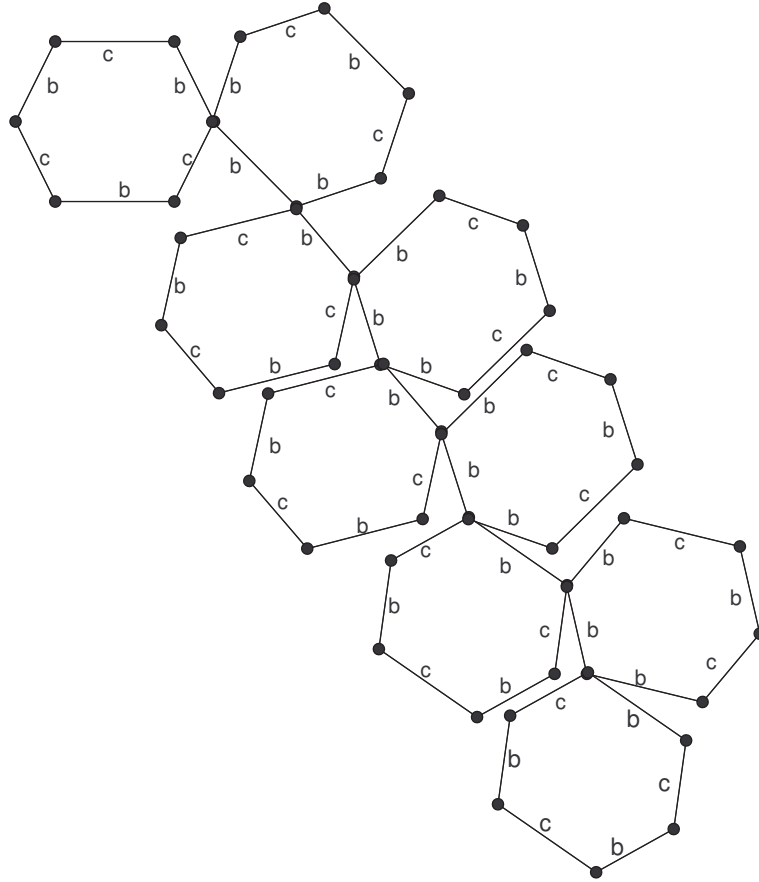


Figure 6: An a -sum V_4 magic labeling of $\diamond[C_6]_9$

Proof. Observe that $\diamond[C_n]_m$ has $mn-m+1$ vertices. If $\diamond[C_n]_m \in \mathcal{V}_a$, then we have $[m(n-1)+1]a = 0$. This implies that $m(n-1)$ is odd. Consequently, m is odd and n even.

Conversely, assume that n is even and m is odd. Consider m copies of C_n . Let the vertices of the i^{th} cycle C_n^i be $(u_1^i, u_2^i, \dots, u_n^i)$. First, consider the pairs (C_n^1, u_1^1) and (C_n^2, u_1^2) and construct $G_1 = (C_n^1, u_1^1) \diamond (C_n^2, u_1^2)$. Next consider the pairs (G_1, u_2^1) and (C_n^3, u_1^3) and construct $G_2 = (G_1, u_2^1) \diamond (C_n^3, u_1^3)$. Proceeding like this, we finally arrive at $G_{m-1} = (G_{m-2}, u_2^{m-1}) \diamond (C_n^m, u_1^m)$. We need to show that $G = G_1 \diamond G_2 \diamond \dots \diamond G_{m-1} \in \mathcal{V}_a$. We label the edges of G by the following table:

$i \backslash \text{edge}$	$u_1^i u_2^i$	$u_2^i u_3^i$	$u_3^i u_4^i$	$u_4^i u_5^i$	$u_5^i u_6^i$	$u_6^i u_7^i$	$\dots$	$u_{n-1}^i u_n^i$	$u_n^i u_1^i$
1	b	c	b	c	b	c	$\dots$	b	c
2	b	b	c	b	c	b	$\dots$	c	b
3	b	c	b	c	b	c	$\dots$	b	c
4	b	b	c	b	c	b	$\dots$	c	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
m	b	c	b	c	b	c	$\dots$	b	c

One can easily verify that this is a a -sum V_4 magic labeling of G . This completes the proof. An a -sum V_4 magic labeling of $\diamond[C_6]_9$ is shown in figure 6. $\square$

Theorem 2.23. $C_m \diamond C_n \in \mathcal{V}_0$ for all m and n .

Proof. Label all edges by a , we obtain $\ell^+ \equiv 0$. $\square$

Theorem 2.24. If $m+n$ is odd, then $C_m \diamond C_n \in \mathcal{V}_{a,0}$.

Proof. Proof follows from 2.21 and 2.23. $\square$

Theorem 2.25. $\diamond C_n \in \mathcal{V}_0$.

Proof. If we label all edges of $\diamond C_n$ by a , we obtain a zero sum V_4 magic labeling of $\diamond C_n$. $\square$

Definition 2.6. A wheel graph denoted by W_n is defined as $W_n \simeq C_n + K_1$, where C_n for $n \geq 3$ is a cycle of length n [4].

Here we need the following lemma.

Lemma 2.2. If $\ell : E(W_1) \rightarrow V_4 \setminus \{0\}$ is a labeling of W_n , then

$$\sum_{i=1}^n \ell^+(u_i) = \ell^+(u) \quad (2.1)$$

where $u_1, u_2, \dots, u_n$ are the vertices of the cycle C_n and u is the central vertex of W_n .

Proof. Observe that

$$\ell^+(u) = \sum_{i=1}^n \ell(uu_i), \quad (2.2)$$

and

$$\ell^+(u_i) = \ell(u_{i-1}u_i) + \ell(u_i u_{i+1}) + \ell(uu_i). \quad (2.3)$$

Therefore

$$\sum_{i=1}^n \ell^+(u_i) = 2 \sum_{i=1}^n \ell(u_i u_{i+1}) + \sum_{i=1}^n \ell(uu_i)$$

That is,

$$\sum_{i=1}^n \ell^+(u_i) - \ell^+(u) = 2 \sum_{i=1}^n \ell(u_i u_{i+1}). \quad (2.4)$$

Note that $\sum_{i=1}^n \ell(u_i u_{i+1}) \in V_4$. Therefore, $2 \sum_{i=1}^n \ell(u_i u_{i+1}) = 0$. Hence equation (2.4) reduces to

$$\sum_{i=1}^n \ell^+(u_i) = \ell^+(u).$$

This completes the proof. $\square$

Theorem 2.26. $W_n \in \mathcal{V}_a$ if and only if n is odd.

Proof. Suppose W_n admits an a -sum V_4 magic labeling. Then by lemma, we have

$$na = a, \quad a \neq 0.$$

This implies that n is odd.

Conversely, assume that n is odd. We will prove that W_n admits an a -sum V_4 magic labeling. Let $\ell : E(W_n) \rightarrow V_4 \setminus \{0\}$ be a labeling of W_n such that $\ell(uu_i) = a$ for all i . Since n is odd, $\sum_{i=1}^n \ell(uu_i) = a$. Thus $\ell^+(u) = a$. Note that $\ell(u_i u_{i+1}) \in V_4$ for $i = 1, 2, \dots, n$, where $u_{n+1} = u_1$. Therefore, $2 \sum_{i=1}^n \ell(u_i u_{i+1}) = 0$. This implies that $\sum_{i=1}^n \ell(u_i u_{i+1}) = 0, a, b$ or c . Without loss of generality assume that $\sum_{i=1}^n \ell(u_i u_{i+1}) = 0$. The other cases are similar. Note that $\sum_{i=1}^n \ell(u_i u_{i+1}) = 0$ can be written as:

$$\ell(u_1 u_2) + \sum_{i=2}^n \ell(u_i u_{i+1}) = 0. \quad (2.5)$$

Let us take $\ell(u_1 u_2) = a$. One can assign b or c to $\ell(u_1 u_2)$ instead of a . If $\ell(u_1 u_2) = a$, the second term in equation (2.5) can be taken as a . That is,

$$\sum_{i=2}^n \ell(u_i u_{i+1}) = a. \quad (2.6)$$

Note that equation (2.6) can be written as:

$$\ell(u_2 u_3) + \sum_{i=3}^n \ell(u_i u_{i+1}) = a. \quad (2.7)$$

For an a -sum V_4 magic graph, we need

$$\ell(u_1 u_2) + \ell(u_2 u_3) + \ell(uu_2) = a.$$

This equation implies that $\ell(u_2 u_3) = a$. Hence $\ell^+(u_2) = a$. From equation (2.7), we have $\sum_{i=3}^n \ell(u_i u_{i+1}) = 0$. That is,

$$\sum_{i=3}^n \ell(u_i u_{i+1}) = 0 \quad (2.8)$$

Again, equation (2.8) can be written as:

$$\ell(u_3 u_4) + \sum_{i=4}^n \ell(u_i u_{i+1}) = 0. \quad (2.9)$$

For an a -sum V_4 magic graph, we need

$$\ell(u_3 u_4) + \ell(uu_3) + \ell(u_2 u_3) = a$$

This implies that $\ell(u_3 u_4) = a$. Hence $\ell^+(u_3) = a$. If we continue this process we finally arrive at $\ell(u_n u_1) = a$ and $\ell^+(u_1) = a$. Thus ℓ is an a -sum V_4 magic labeling of W_n . $\square$

A step by step procedure for finding an a -sum magic map for W_n , when n is odd is given below:

1. For $i = 1, 2, \dots$, setting

$$\ell(uu_i) = a \text{ or } b \text{ or } c.$$

2. Consider the equation $\sum_{i=1}^n \ell(u_i u_{i+1}) = 0$. Assume that $\ell(uu_i) = a$.
3. Split 0 into two parts. We have the following possibilities:

$$a + a = 0, \quad b + b = 0, \quad c + c = 0$$

4. Consider the first sum $a + a = 0$ and take $\ell(u_1 u_2)$ as a . Then $\sum_{i=2}^n \ell(u_i u_{i+1}) = a$. One can consider the other two cases also.
5. Split the summation $\sum_{i=2}^n \ell(u_i u_{i+1}) = a$ in the following form

$$\ell(u_2 u_3) + \sum_{i=3}^n \ell(u_i u_{i+1}) = a.$$

Find the value to $\ell(u_2 u_3)$ from the following equation:

$$\ell(u_1 u_2) + \ell(uu_2) + \ell(u_2 u_3) = a.$$

6. Continue this processes up to the $(n-1)^{\text{th}}$ step. Finally the value of $\ell(u_n u_1)$ is determined by the equation:

$$\ell(u_{n-1} u_n) + \ell(uu_n) + \ell(u_1 u_n) = a$$

Observe that a -sum V_4 magic labeling of W_n is not unique. The following is another procedure for obtaining an a -sum V_4 magic labeling of W_n .

1. Consider the equation

$$\sum_{i=1}^n \ell(u_i u_{i+1}) = 0, a, b \text{ or } c. \quad (2.10)$$

Without loss of generality assume that

$$\sum_{i=1}^n \ell(u_i u_{i+1}) = 0. \quad (2.11)$$

The equation (2.11) can be written as:

$$\ell(u_1 u_2) + \sum_{i=2}^n \ell(u_i u_{i+1}) = 0. \quad (2.12)$$

Assign a or b or c to $\ell(u_1 u_2)$. Let us assign a to $\ell(u_1 u_2)$. Then from equation (2.12) one obtain,

$$\sum_{i=2}^n \ell(u_i u_{i+1}) = a. \quad (2.13)$$

Equation (2.13) can be written as:

$$\ell(u_2 u_3) + \sum_{i=3}^n \ell(u_i u_{i+1}) = a. \quad (2.14)$$

Assign any value to $\ell(u_2 u_3)$ from the set $\{a, b, c\}$. Let us assume that $\ell(u_2 u_3) = b$. Choose $\ell(uu_2)$ such that

$$\ell(u_1 u_2) + \ell(uu_2) + \ell(u_2 u_3) = a. \quad (2.15)$$

Hence we have

$$\ell^+(u_2) = a.$$

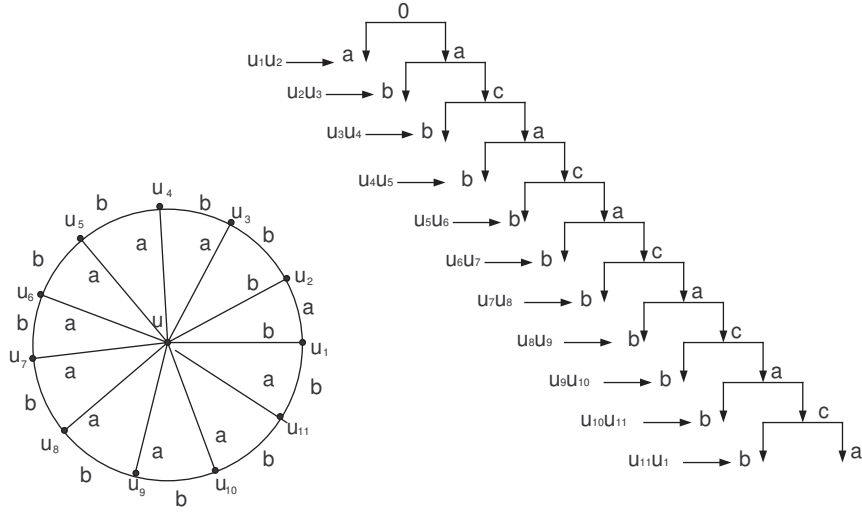


Figure 7: An a -sum V_4 magic labeling of W_{11}

2. From equation (2.13), we have

$$\sum_{i=3}^n \ell(u_i u_{i+1}) = c \quad (2.16)$$

Applying the same procedure as explained above, one obtain:

$$\ell^+(u_3) = a.$$

3. Continue the above processes. Finally, we obtain

$$\ell^+(u_1) = a.$$

4. Since ℓ is a labeling of W_n , by lemma 2.2, we have

$$\ell^+(u) = \sum_{i=1}^n \ell^+(u_i) = na$$

Since n is odd, we have $na = a$. Therefore, we have $\ell^+(u) = a$.

Several a -sum magic labelings of W_{11} are shown in figure 7 and 8.

Theorem 2.27. $W_n \in \mathcal{V}_0$, if n is odd.

Proof. Suppose n is odd. Define $\ell : E(W_n) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(wu_i) &= a \text{ for } i = 1, 2, \dots, n-2, \\ \ell(uu_i) &= c \text{ for } i = n-1, \\ \ell(uu_i) &= b \text{ for } i = n, \\ \ell(u_i u_{i+1}) &= b \text{ for } i = 1, 3, \dots, n-2 \end{aligned}$$

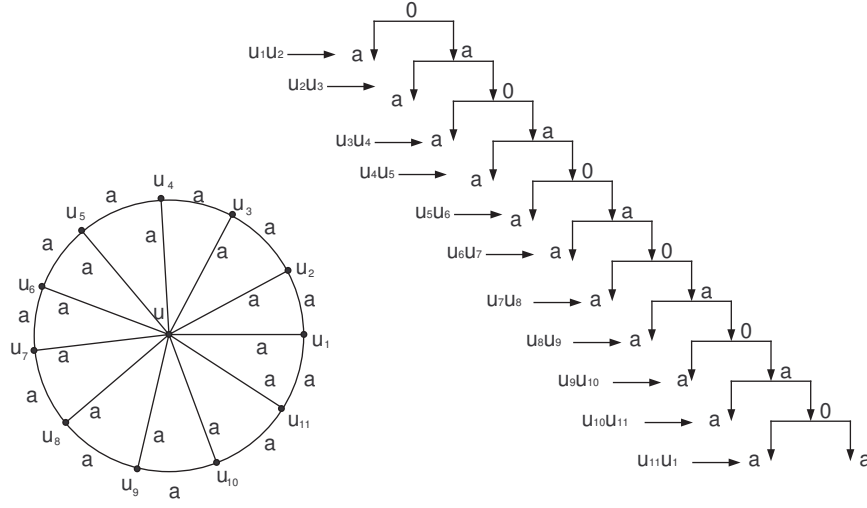


Figure 8: An a -sum V_4 magic labeling of W_{11}

$$\begin{aligned}\ell(u_i u_{i+1}) &= c \text{ for } i = 2, 4, \dots, n-3 \\ \ell(u_{n-1} u_n) &= a \\ \ell(u_n u_1) &= c.\end{aligned}$$

Obviously ℓ is a zero sum V_4 magic labeling of W_n . □

A zero sum magic labeling of W_7 and W_{13} are shown in figure 9. Further, two different zero sum V_4 magic labeling of W_{11} is shown in figure 10 and figure 11.

Theorem 2.28. $W_n \in \mathcal{V}_0$, if n is even

Proof. Let $\ell : E(W_n) \rightarrow V_4 \setminus \{0\}$ be a labeling of W_n . Then $2 \sum_{i=1}^n \ell(u_i u_{i+1}) = 0$. This implies that $\sum_{i=1}^n \ell(u_i u_{i+1}) = 0, a, b$ or c . Without loss of generality assume that

$$\sum_{i=1}^n \ell(u_i u_{i+1}) = 0. \quad (2.17)$$

Rest of the proof is exactly similar to the algorithm for finding the a -sum V_4 magic labeling of W_n explained above subject to the condition that no element will repeat consecutively on the outer circle of W_n . □

Theorem 2.29. If $n \equiv 0 \pmod{3}$, then W_n admits a zero sum V_4 magic labeling.

Proof. Define $\ell : E(W_n) \rightarrow V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}\ell(u_i u_{i+1}) &= b, \ell(u u_i) = c & \text{for } i = 1, 4, 7, \dots, n-2, \\ \ell(u_i u_{j+1}) &= c, \ell(u u_i) = a & \text{for } j = 2, 5, 8, \dots, n-1, \\ \ell(u_i u_{i+1 \pmod{n}}) &= a, \ell(u u_i) = b & \text{for } i = 3, 6, 9, \dots, n.\end{aligned}$$

Obviously ℓ is a zero sum magic labeling of W_n . □

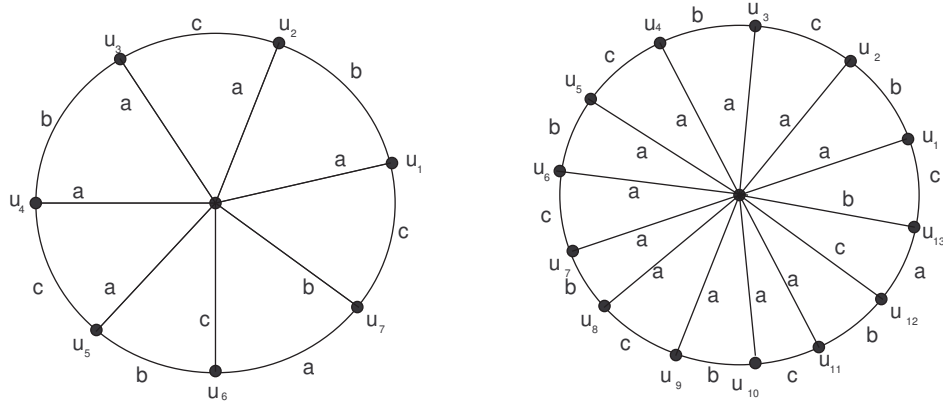


Figure 9: A Zero-sum V_4 magic labeling of W_7 and W_{13}

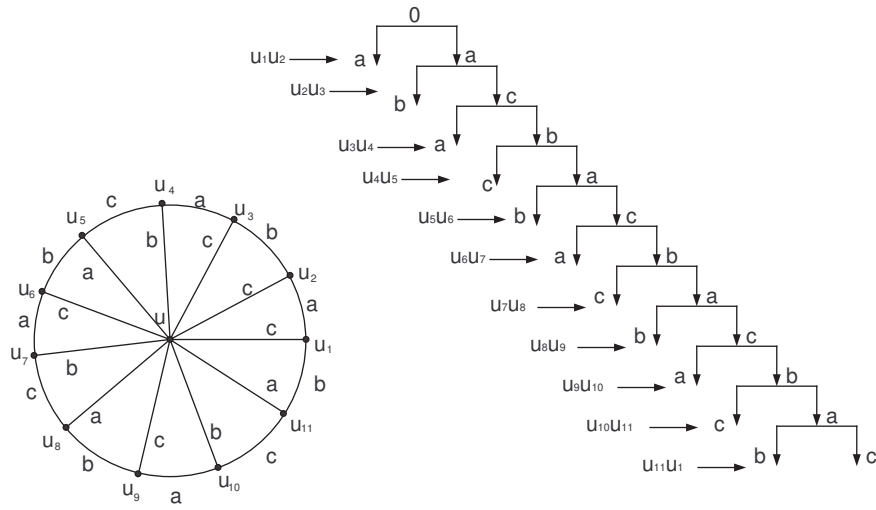


Figure 10: A zero-sum V_4 magic labeling of W_{11}

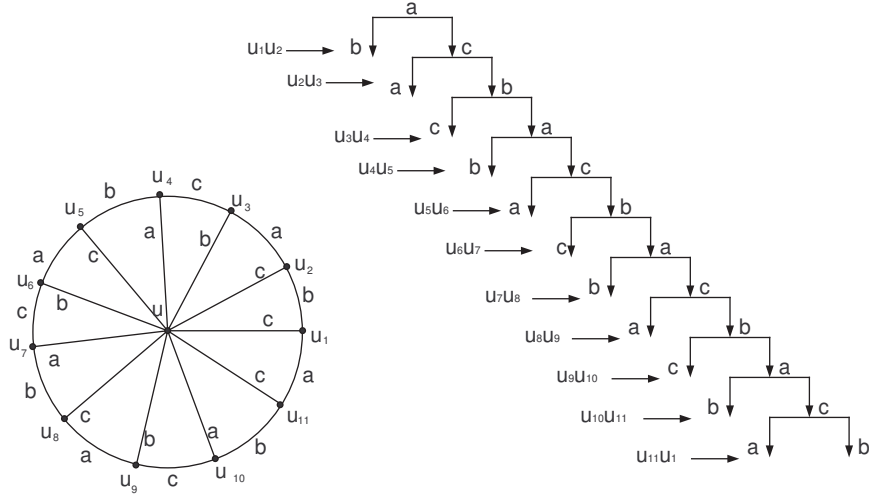


Figure 11: A 0-sum V_4 magic labeling of W_{11}

Theorem 2.30. *If W_n is a -sum V_4 magic and if k is odd, then W_{nk} is a -sum V_4 magic.*

Proof. Assume that W_n is a -sum V_4 magic. Then by theorem 2.26, we have n is odd. Since k is odd this implies that nk is odd. Hence theorem 2.26 tells us that W_{nk} is a -sum V_4 magic. $\square$

Next, we will explain a procedure for obtaining an a -sum V_4 magic labeling W_{nk} if an a -sum V_4 magic labeling of W_n is known.

Let $C_{n,1} : v_1, v_2, v_3, \dots, v_n, v_1$ and v be the center vertex of W_n . Let $C_{nk,1} : u_1, u_2, \dots, u_{kn}, u_1$ and u be the center vertex of $W_{nk,1}$. Let $\ell : E(W_n) \rightarrow V_4 \setminus \{0\}$ be an a -sum V_4 magic labeling of W_n . Whenever $m \equiv i(\text{mod } n)$, define a function $\ell' : E(W_{nk}) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}\ell'(uu_m) &= \ell(vv_i), \text{ for } m \equiv i(\text{mod } n) \\ \ell'(u_mu_{m+1}) &= \ell(v_iv_{i+1}), \text{ for } m \equiv i(\text{mod } n).\end{aligned}$$

If ℓ'^+ is the induced vertex labeling of W_{nk} , then $\ell'^+(u) = k\ell^+(v) = ka = a$ and

$$\begin{aligned}\ell'^+(u_i) &= \ell'(u_{m-1}u_m) + \ell'(uu_m) + \ell'(u_mu_{m+1}) \\ &= \ell(u_{(m-1) \bmod n}u_{m \bmod n}) + \ell(uu_{m \bmod n}) + \ell(u_{m \bmod n}u_{(m+1) \bmod n}) = a.\end{aligned}$$

Hence ℓ' is an a -sum V_4 magic labeling of W_{nk} . An a -sum V_4 magic labeling of W_3 and W_{15} is shown in figure 12.

Theorem 2.31. *If W_n is zero-sum V_4 magic, so is W_{kn} for every $k \geq 2$.*

Definition 2.7. A double-wheel graph $W_{n,2}$ can be obtained as join of $2C_n + K_1$, and inductively we can construct an m -level wheel graph denoted by $W_{n,m}$ as follows $W_{n,m} \simeq mC_n + K_1$ [4].

Let $C_{n,1}, \dots, C_{n,m}$ represent the cycles of $W_{n,m}$ at levels $1, \dots, m$, respectively, as shown in figure 13.

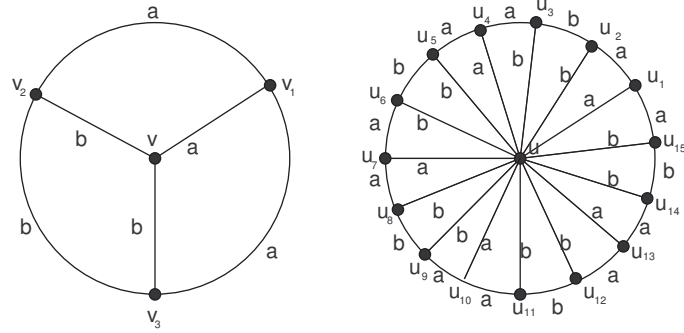


Figure 12: An α -sum V_4 magic labeling of W_3 and W_{15}

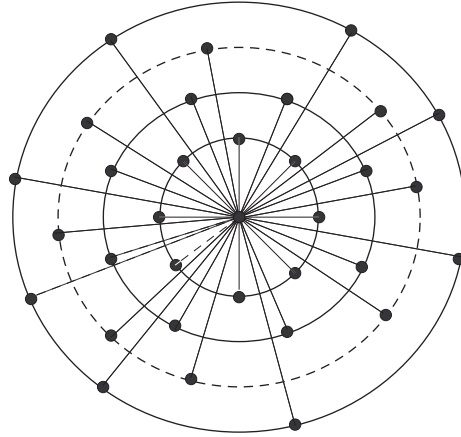


Figure 13: An m -level Wheel: $W_{n,m}$

Lemma 2.3. *If $\ell : E(W_{n,m}) \rightarrow V_4 \setminus \{0\}$ is a labeling of $W_{n,m}$, then*

$$\sum_{j=1}^m \sum_{i=1}^n \ell^+(u_{i,j}) = \ell^+(u) \quad (2.18)$$

where $u_{1,j}, u_{2,j}, \dots, u_{n,j}, u_{1,j}$ are the vertices of the cycle $C_{n,j}$ and u is the central vertex of $W_{n,m}$.

Theorem 2.32. $W_{n,m} \in \mathcal{V}_a$ if and only if both m and n are odd.

Proof. If $W_{n,m} \in \mathcal{V}_a$, then by lemma 2.3, we have

$$(mn)a = a.$$

This implies that mn is odd or equivalently m and n are both odd.

Conversely, assume that both m and n are odd. If we label all the edges of $W_{n,m}$ by a , then obviously $\ell^+(u) = a$ and $\ell^+(u_{ij}) = a$. $\square$

Theorem 2.33. $W_{n,m} \in \mathcal{V}_0$ for all m and n .

Proof. Obvious. $\square$

Theorem 2.34. *If $W_{n,m} \in \mathcal{V}_a$, then $W_{kn,m} \in \mathcal{V}_a$ if k is odd.*

Proof. $W_{n,m} \in \mathcal{V}_a$ implies that mn is odd. This implies that both m and n are odd. Now, $W_{kn,m} \in \mathcal{V}_a$ if mnk is odd. This implies that k is odd. $\square$

Theorem 2.35. *If $W_{n,m} \in \mathcal{V}_0$, then $W_{nk,m} \in \mathcal{V}_0$ for any $k \geq 1$.*

Definition 2.8. A subdivided wheel graph denoted by SW_n is obtained by dividing each spoke uu_i . Similarly, we can define the subdivided m -level graph $SW_{n,m}$.

Lemma 2.4. *If $\ell : E(SW_n) \rightarrow V_4 \setminus \{0\}$ is a labeling of SW_n , then*

$$\sum_{i=1}^n \ell^+(u_i) + \sum_{i=1}^n \ell^+(v_i) = \ell^+(u). \quad (2.19)$$

where $u_1, u_2, \dots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \dots, v_n$ are the vertices corresponding the subdivision of the spokes uu_i and u is the central vertex of W_n .

Theorem 2.36. $SW_n \notin \mathcal{V}_a$ for any $n \geq 3$.

Proof. Assume that SW_n admits an a -sum V_4 magic labeling. Then by lemma 2.4, we have

$$na + na = a$$

This implies that $a = 0$. $\square$

Theorem 2.37. $SW_n \in \mathcal{V}_0$ for any $n \geq 3$.

Proof. We consider two cases:

Case 1: If n is even, define $\ell : E(SW_n) \rightarrow V_4 \setminus \{0\}$ as follows:

$$\begin{aligned} \ell(uv_i) &= a, \text{ for } i = 1, 2, 3, \dots, n, \\ \ell(v_i u_i) &= a, \text{ for } i = 1, 2, 3, \dots, n, \\ \ell(u_i u_{(i+1)}) &= b, \text{ for } i = 1, 3, \dots, n-1, \\ \ell(u_i u_{(i+1) \pmod n}) &= c, \text{ for } i = 2, 4, \dots, n. \end{aligned}$$

Obviously $\ell^+(u) = \ell^+(u_i) = \ell^+(v_i) = 0$. Hence $SW_n \in \mathcal{V}_0$ if n is even.

Case 2: If n is odd, define $\ell : E(SW_n) \rightarrow V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
\ell(uv_i) &= a, \text{ for } i = 1, 2, 3, \dots, n-2, \\
\ell(uv_i) &= b, \text{ for } i = n-1, \\
\ell(uv_i) &= c, \text{ for } i = n, \\
\ell(v_iu_i) &= a, \text{ for } i = 1, 2, \dots, n-2, \\
\ell(v_iu_i) &= b, \text{ for } i = n-1, \\
\ell(v_iu_i) &= c, \text{ for } i = n, \\
\ell(u_iu_{i+1}) &= b, \text{ for } i = 2, 4, \dots, n-3, \\
\ell(u_iu_{i+1}) &= c, \text{ for } i = 1, 3, \dots, n-2, \\
\ell(u_{n-1}u_n) &= a, \\
\ell(u_nu_1) &= b.
\end{aligned}$$

Observe that $\ell^+(u) = \ell^+(u_i) = \ell^+(v_i) = 0$. Hence $SW_n \in \mathcal{V}_0$ if n is odd. □

Lemma 2.5. If $\ell : E(SW_{n,m}) \rightarrow V_4 \setminus \{0\}$ is a labeling of $SW_{n,m}$, then

$$\sum_{j=1}^m \sum_{i=1}^n \ell^+(u_{i,j}) + \sum_{j=1}^m \sum_{i=1}^n \ell^+(v_{i,j}) = \ell^+(u). \quad (2.20)$$

where $u_{1,j}, u_{2,j}, \dots, u_{n,j}, u_{1,j}$ are the vertices of the cycle $C_{n,j}$, $v_{1,j}, v_{2,j}, \dots, v_{n,j}$ are the subdivisions corresponding to the edges $uu_{i,j}$ and u is the central vertex.

Theorem 2.38. $SW_{n,m} \notin \mathcal{V}_a$ for any n and m .

Proof. Obvious. □

Theorem 2.39. $SW_{n,m} \in \mathcal{V}_0$ for any n and m .

Proof. We consider two cases:

Case 1: If n is even, for $j = 1, 2, \dots, m$, define $\ell : E(SW_{n,m}) \rightarrow V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
\ell(uv_{i,j}) &= a \text{ for } i = 1, 2, 3, \dots, n, \\
\ell(v_{i,j}u_{i,j}) &= a \text{ for } i = 1, 2, 3, \dots, n, \\
\ell(u_{i,j}u_{i+1,j}) &= b, \text{ for } i = 1, 3, \dots, n-1, \\
\ell(u_{i,j}u_{i+1,j}) &= c, \text{ for } i = 2, 4, \dots, n.
\end{aligned}$$

Obviously ℓ is a zero-sum magic labeling of $SW_{n,m}$.

Case 2: If n is odd, for $j = 1, 2, \dots, m$, define $\ell : E(SW_{n,m}) \rightarrow V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
\ell(uv_{i,j}) &= a, \text{ for } i = 1, 2, 3, \dots, n-2, \\
\ell(uv_{i,j}) &= b, \text{ for } i = n-1 \\
\ell(uv_{i,j}) &= c, \text{ for } i = n \\
\ell(v_{i,j}u_{i,j}) &= a, \text{ for } i = 1, 2, 3, \dots, n-2, \\
\ell(u_{i,j}v_{i,j}) &= b, \text{ for } i = n-1, \\
\ell(u_{i,j}v_{i,j}) &= c, \text{ for } i = n,
\end{aligned}$$

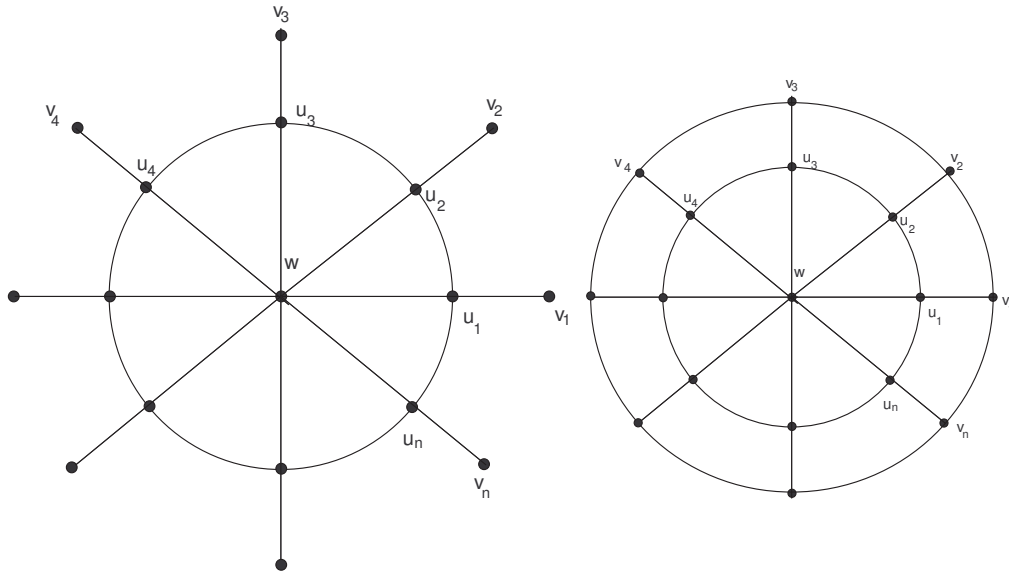


Figure 14: The helm: H_n (left) and the closed helm $:H(2, n)$ (right)

$$\begin{aligned}\ell(u_{i,j}u_{i+1,j}) &= c, \text{ for } i = 1, 3, \dots, n-2 \\ \ell(u_{i,j}u_{i+1,j}) &= b, \text{ for } i = 2, 4, \dots, n-3 \\ \ell(u_{i,j}u_{n,j}) &= a, \text{ for } i = n-1 \\ \ell(u_{i,j}u_{1,j}) &= b, \text{ for } i = n.\end{aligned}$$

Obviously ℓ is a zero-sum magic labeling of $SW_{n,m}$.

□

Definition 2.9. The helm H_n is the graph obtained from the wheel W_n by attaching a pendant edge at each vertex of the cycle C_n (see figure 14)[5].

Lemma 2.6. If $\ell : E(H_n) \rightarrow V_4 \setminus \{0\}$ is a labeling of H_n , then

$$\sum_{i=1}^n \ell^+(u_i) + \sum_{i=1}^n \ell^+(v_i) = \ell^+(u). \quad (2.21)$$

where $u_1, u_2, \dots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \dots, v_n$ are the pendant vertices corresponding to the spokes uu_i and u is the central vertex of W_n .

Theorem 2.40. $H_n \notin \mathcal{V}_a$ for any n .

Proof. Proof follows from lemma 2.6.

□

Theorem 2.41. $H_n \notin \mathcal{V}_0$ for any n .

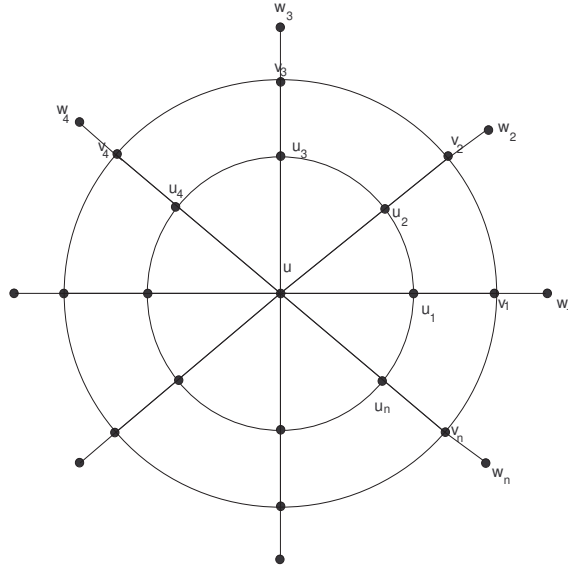


Figure 15: Web graph : $W(2, n)$

Proof. Obvious. □

Definition 2.10. The web graph $W(2, n)$ is the graph obtained by joining the pendant points of a helm H_n to form a cycle and then adding a single pendant edge to each vertex of the outer cycle (see figure 15)[5].

Lemma 2.7. If $\ell : E(W(2, n)) \rightarrow V_4 \setminus \{0\}$ is a labeling of $W(2, n)$, then

$$\sum_{i=1}^n \ell^+(u_i) + \sum_{i=1}^n \ell^+(v_i) + \sum_{i=1}^n \ell^+(w_i) = \ell^+(u). \quad (2.22)$$

where $u_1, u_2, \dots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \dots, v_n$ are the vertices of $C_{n,2}$, $w_1, w_2, \dots, w_n$ are the pendant vertices and u is the hub of $W(2, n)$.

Theorem 2.42. $W(2, n) \in \mathcal{V}_a$ if and only if n is odd.

Proof. Assume that $W(2, n) \in \mathcal{V}_a$. Then from lemma 2.7, we have

$$na + na + na = a$$

This implies that $na = a$. This equation holds if and only if n is odd.

Conversely, assume that n is odd. Define a mapping $\ell : E(W(2, n)) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(uu_i) &= a \text{ for } i = 1, 2, \dots, n, \\ \ell(u_iu_{i+1}) &= b \text{ for } i = 1, 3, \dots, n, \\ \ell(u_nu_1) &= a \end{aligned}$$

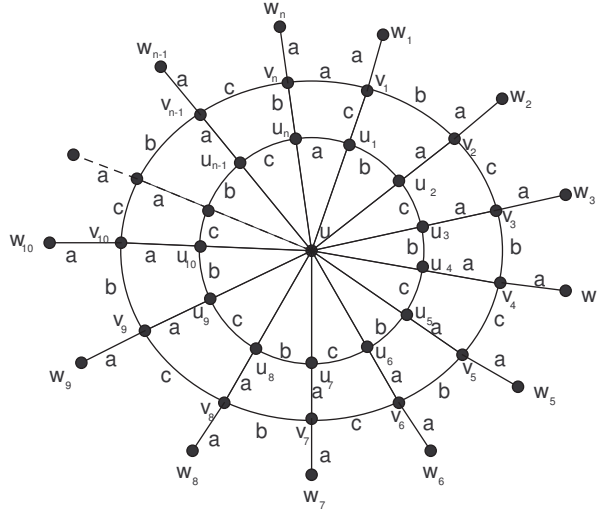


Figure 16: An a -sum V_4 magic labeling of $W(2, n)$

$$\begin{aligned}
 \ell(u_i u_{i+1}) &= c \text{ for } i = 2, 4, \dots, n-1, \\
 \ell(v_i v_{i+1}) &= b \text{ for } i = 1, 3, \dots, n, \\
 \ell(v_n v_1) &= a \\
 \ell(v_i v_{i+1}) &= c \text{ for } i = 2, 4, \dots, n-1, \\
 \ell(u_1 v_1) &= c, \\
 \ell(u_i v_i) &= a \text{ for } i = 2, 4, \dots, n-1, \\
 \ell(u_n v_n) &= b, \\
 \ell(v_i w_i) &= a.
 \end{aligned}$$

Obviously, $\ell^+(u) = a$ and $\ell^+(u_i) = \ell^+(v_i) = \ell(w_i) = a$ for $i = 1, 2, \dots, n$. A a -sum V_4 magic labeling is shown in figure 16.

□

Theorem 2.43. $W(2, n) \notin \mathcal{V}_0$ for any n .

Proof. Obvious.

□

Definition 2.11. The generalized web graph $W(t, n)$ is the graph obtained by iterating the processes of constructing web graph $W(2, n)$ from the helm H_n , so that the web has t n -cycles (See figure 17)[5].

Lemma 2.8. If $\ell : E(W(t, n)) \rightarrow V_4 \setminus \{0\}$ is a labeling of $W(t, n)$, then

$$\sum_{i=1}^t \sum_{j=1}^n \ell^+(u_{i,j}) + \sum_{i=1}^n \ell^+(v_i) = \ell^+(u). \quad (2.23)$$

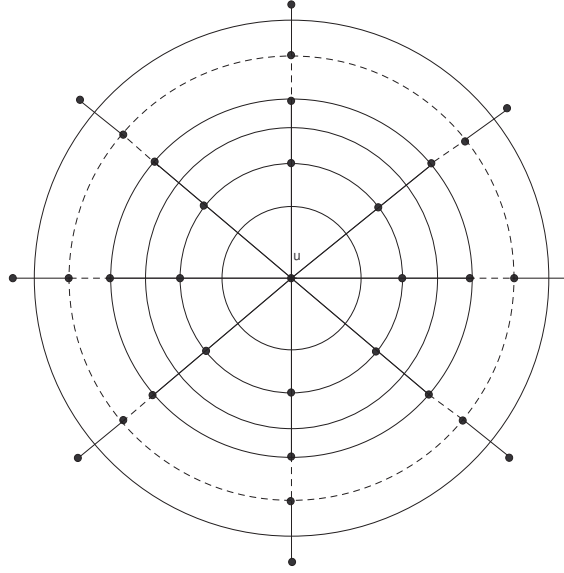


Figure 17: Generalised Web graph : $W(t, n)$,

where $u_{1,j}, u_{2,j}, \dots, u_{n,j}$ are the vertices of the cycle $C_{n,j}$, $j = 1, 2, \dots, t$ and $v_1, v_2, \dots, v_n$ are the pendant vertices and u is the hub of $W(2, n)$.

Proof. Obvious. □

Theorem 2.44. $W(t, n) \in \mathcal{V}_a$ if and only if n is odd and t is even.

Proof. Assume that $W(t, n) \in \mathcal{V}_a$. Then by lemma 2.8, we have

$$n(t+1)a = a$$

This implies that n is odd and t is even. Conversely, if n is odd and t is even one can easily prove that $W(t, n) \in \mathcal{V}_a$. □

Theorem 2.45. $W(t, n) \notin \mathcal{V}_0$ for any n and any t .

Proof. Obvious. □

Definition 2.12. The generalized web graph without center, $W_0(t, n)$ is the graph obtained by removing the central vertex of $W(t, n)$ [5]. The graph of $W_0(t, n)$ is shown in figure 18.

Lemma 2.9. If $\ell : E(W_0(t, n)) \rightarrow V_4 \setminus \{0\}$ is a labeling of $W_0(t, n)$, then

$$\sum_{j=1}^t \sum_{i=1}^n \ell^+(u_{i,j}) + \sum_{i=1}^n \ell^+(v_i) = 0. \quad (2.24)$$

where $u_{1,j}, u_{2,j}, \dots, u_{n,j}$ are the vertices of the cycle $C_{n,j}$, and $v_1, v_2, \dots, v_n$ are the pendant vertices.

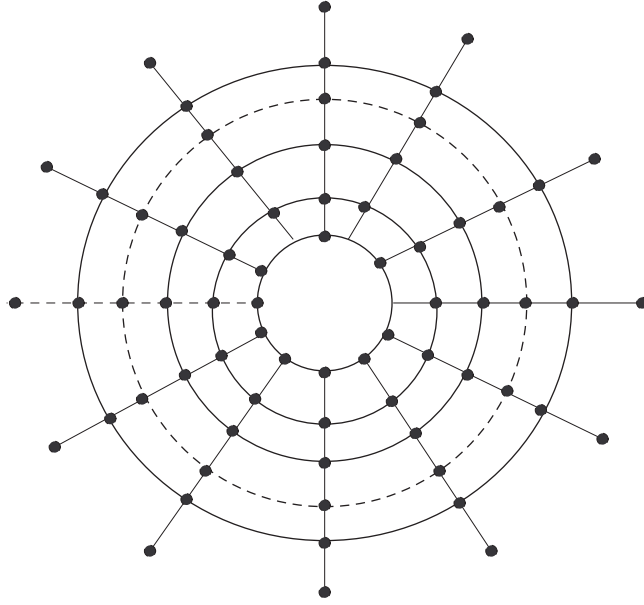


Figure 18: Generalised Web graph without centre: $W_0(t, n)$

Proof. Obvious. □

Theorem 2.46. *If $W_0(t, n) \in \mathcal{V}_a$ if and only if $n(t+1)$ is even.*

Proof. First, assume that $W_0(t, n) \in \mathcal{V}_a$. Then by lemma 2.9, we have $nta + na = 0$. This implies that $n(t+1)$ is even.

Conversely, assume that $n(t+1)$ is even. we consider the following cases:

Case 1: If n and t are even, define $\ell : E(W_0(t, n)) \rightarrow V_4 \setminus \{0\}$ as follows:

$$\begin{aligned} & \ell(u_{i,1}u_{(i+1)(\bmod n),1}) = a \text{ for } i = 1, 2, 3, \dots, n \\ & \text{for } j = 2, 3, \dots, t : \\ & \quad \ell(u_{i,j}u_{i+1,j}) = c \text{ for } i = 1, 3, \dots, n-1 \\ & \quad \ell(u_{i,j}u_{(i+1)(\bmod n),j}) = b \text{ for } i = 2, 4, \dots, n \\ & \text{end for} \\ & \text{for } j = 1, 2, \dots, n : \\ & \quad \ell(u_{i,j}u_{i,j+1}) = a \text{ for } i = 1, 2, \dots, t-1 \\ & \text{end for} \\ & \ell(u_{i,t}v_i) = a \text{ for } i = 1, 2, 3, 4, \dots, n. \end{aligned}$$

Case 2: Assume that n is even and t is odd. In this case the labeling is exactly similar to Case 1.

Case 3: If n is odd and t is odd, define $\ell : E(W_0(t, n)) \rightarrow V_4 \setminus \{0\}$ as follows:

$$\ell(u_{i,1}u_{(i+1)(\bmod n),1}) = a \text{ for } i = 1, 2, 3, \dots, n$$

for $j = 2, 3, \dots, t$:
 $\ell(u_{i,j}u_{i+1,j}) = b$ for $i = 1, 3, \dots, n-2$
 $\ell(u_{i,j}u_{(i+1)(\bmod n),j}) = c$ for $i = 2, 4, \dots, n-1$
 end for
 $\ell(u_{n,i}u_{1,i}) = a$ for $i = 1, 2, 3, 4, \dots, n$,
 $\ell(u_{i,t}v_i) = a$ for $i = 1, 2, 3, 4, \dots, n$,
 for $k = 2, 3, \dots, n-1$:
 $\ell(u_{k,i}u_{k,i+1}) = a$, for $i = 1, 2, 3, \dots, t-1$,
 end for
 $\ell(u_{n,j}u_{1,j}) = a$, for $j = 1, 2, 3, \dots, t-1$,
 $\ell(u_{1,i}u_{1,i+1}) = a$, for $i = 1, 3, \dots, t-2$,
 $\ell(u_{1,i}u_{1,i+1}) = c$, for $i = 2, 4, \dots, t-1$,
 $\ell(u_{n,i}u_{n,i+1}) = a$, for $i = 1, 3, \dots, t-2$,
 $\ell(u_{n,i}u_{n,i+1}) = b$, for $i = 2, 4, \dots, t-1$.

Obviously $\ell^+(u_{i,j}) = a$ and $\ell^+(v_i) = a$.

□

Theorem 2.47. $W_0(t, n) \notin \mathcal{V}_0$ for any n and t .

Proof. Obvious.

□

Definition 2.13. A closed helm $H(2, n)$ is the graph obtained from a helm by joining each pendant vertex to form a cycle [6]. A closed helm $H(2, n)$ is shown in figure 14.

Lemma 2.10. If $\ell : E(H(2, n)) \rightarrow V_4 \setminus \{0\}$ is a labeling of $H(2, n)$, then

$$\sum_{i=1}^n \ell^+(u_i) + \sum_{i=1}^n \ell^+(v_i) = \ell^+(w) \quad (2.25)$$

where $u_1, u_2, \dots, u_n$ are the vertices of the cycle $C_{n,1}$, $v_1, v_2, \dots, v_n$ are the vertices of the cycle $C_{n,2}$ and w is the central vertex.

Proof. Obvious.

□

Theorem 2.48. $H(2, n) \notin \mathcal{V}_a$ for any n .

Proof. Assume that $H(2, n) \in \mathcal{V}_a$. Then by lemma 2.10, we have $na + na = a$. This implies that $a = 0$. This is a contradiction. □

Theorem 2.49. $H(2, n) \in \mathcal{V}_0$ for all n .

Proof. Case 1 Assume that n is even. Define a labeling $\ell : E(H(2, n)) \rightarrow V_4 \setminus \{0\}$ as follows:

$$\begin{aligned}
 \ell(u_i w) &= a & \text{for } i &= 1, 2, \dots, n, \\
 \ell(u_i u_{i+1}) &= a & \text{for } i &= 1, 2, \dots, n, \\
 \ell(u_i v_i) &= a & \text{for } i &= 1, 2, \dots, n, \\
 \ell(v_i v_{i+1}) &= b & \text{for } i &= 1, 3, \dots, n-1, \\
 \ell(v_i v_{i+1}) &= c & \text{for } i &= 2, 4, \dots, n.
 \end{aligned}$$

Obviously, ℓ is a zero-sum V_4 magic labeling of $H(2, n)$.

Case 2: Assume that n is odd. Define a labeling $\ell : E(H(2, n)) \rightarrow V_4 \setminus \{0\}$ as follows:

$$\begin{aligned} \ell(u_1w) &= a, \ell(u_2w) = b, \ell(u_3w) = c, \\ \ell(u_iw) &= a, \quad \text{for } i = 4, 5, \dots, n, \\ \ell(u_iu_{i+1}) &= a \quad \text{for } i = 1, 2, \dots, n, \\ \ell(u_1v_1) &= a, \ell(u_2v_2) = b, \ell(u_3v_3) = c, \\ \ell(u_iv_i) &= a, \quad \text{for } i = 4, 5, \dots, n, \\ \ell(v_1v_2) &= c, \ell(v_2v_3) = a, \ell(v_3v_4) = b, \\ \ell(v_iv_{i+1}) &= c \quad \text{for } i = 4, 6, \dots, n-1, \\ \ell(v_iv_{i+1}) &= b \quad \text{for } i = 5, 7, \dots, n. \end{aligned}$$

One can easily verify that ℓ is a zero sum magic labeling of $H(2, n)$. □

Definition 2.14. Closed generalized helms $H(t, n)$ are obtained by taking a generalized web and joining pendent vertices to form a cycle [6].

Lemma 2.11. If $\ell : E(H(t, n)) \rightarrow V_4 \setminus \{0\}$ is a labeling of $H(t, n)$, then

$$\sum_{j=1}^t \sum_{i=1}^n \ell^+(u_{i,j}) = \ell^+(w). \quad (2.26)$$

where $u_{1,j}, u_{2,j}, \dots, u_{n,j}$ are the vertices of the cycle $C_{n,j}$, and w is the central vertex.

Proof. Obvious. □

Theorem 2.50. $H(t, n) \in \mathcal{V}_a$ if and only if both n and t are odd.

Proof. First, assume that $H(t, n) \in \mathcal{V}_a$. Then by lemma 2.11, we have $(nt+1)a = 0$. This implies that both n and t are odd. Conversely, assume that both n and t are odd. Define $\ell : E(H(t, n)) \rightarrow V_4 \setminus \{0\}$ by:

$$\begin{aligned} \ell(u_{1,1}w) &= a, \\ \ell(u_{i,1}w) &= b, \quad \text{for } i = 2, 3, \dots, n, \\ \text{for } j &= 1, 2, \dots, t-1 : \\ \ell(u_{i,j}u_{i+1,j}) &= c, \quad \text{for } i = 1, 3, \dots, n-2, \\ \ell(u_{i,j}u_{i+1,j}) &= b, \quad \text{for } i = 2, 4, \dots, n-1, n \\ \text{end for} \\ \ell(u_{i,t}u_{i+1,t}) &= b, \quad \text{for } i = 1, 3, \dots, n, \\ \ell(u_{i,t}u_{i+1,t}) &= a, \quad \text{for } i = 2, 4, \dots, n-1, \\ \text{for } j &= 1, 2, \dots, t-1 : \\ \ell(u_{1,j}u_{1,j+1}) &= a, \\ \ell(u_{i,j}u_{i,j+1}) &= b, \quad \text{for } i = 2, 3, \dots, n-1, \\ \text{end for} \\ \ell(u_{n,j}u_{n,j+1}) &= c, \quad \text{for } j = 1, 3, \dots, t-2, \\ \ell(u_{n,j}u_{n,j+1}) &= b, \quad \text{for } j = 2, 4, \dots, t-1. \end{aligned}$$

Obviously ℓ is an a -sum magic labeling of $H(t, n)$. □

Theorem 2.51. $H(t, n) \in \mathcal{V}_0$ for all n and t .

Proof. **Case 1:** Assume that n is even. Define $\ell : E(H(t, n)) \rightarrow V_4 \setminus \{0\}$ by:

$$\begin{aligned} \ell(u_{i,1}w) &= a, \quad \text{for } i = 1, 2, \dots, n. \\ \text{for } j = 1, 2, \dots, t-1 : \\ \ell(u_{i,j}u_{i,j+1}) &= a, \quad \text{for } i = 1, 2, \dots, n, \\ \ell(u_{i,j}u_{i+1,j}) &= a, \quad \text{for } i = 1, 2, \dots, n, \\ \text{end for} \\ \ell(u_{i,t}u_{i+1,t}) &= b, \quad \text{for } i = 1, 3, \dots, n-1, \\ \ell(u_{i,t}u_{i+1,t}) &= c, \quad \text{for } i = 2, 4, \dots, n, \end{aligned}$$

Obviously, ℓ is a zero sum V_4 magic labeling of $E(H(t, n))$.

Case 2: Assume that n is odd. Define $\ell : E(H(t, n)) \rightarrow V_4 \setminus \{0\}$ by:

$$\begin{aligned} \ell(u_{1,1}w) &= a, \quad \ell(u_{2,1}w) = b, \quad \ell(u_{3,1}w) = c \\ \ell(u_{i,1}w) &= a, \quad \text{for } i = 4, 5, \dots, n, \\ \text{for } j = 1, 2, \dots, t-1 : \\ \ell(u_{i,j}u_{i+1,j}) &= a, \quad \text{for } i = 1, 2, \dots, n, \\ \ell(u_{1,j}u_{1,j+1}) &= a, \\ \ell(u_{2,j}u_{2,j+1}) &= b, \\ \ell(u_{3,j}u_{3,j+1}) &= c, \\ \ell(u_{i,j}u_{i,j+1}) &= a, \quad \text{for } i = 4, 5, \dots, n, \\ \text{end for} \\ \ell(u_{1,t}u_{2,t}) &= c, \\ \ell(u_{2,t}u_{3,t}) &= a, \\ \ell(u_{3,t}u_{4,t}) &= b, \\ \ell(u_{i,t}u_{i+1,t}) &= c, \quad \text{for } i = 4, 6, \dots, n-1, \\ \ell(u_{i,t}u_{i+1,t}) &= b, \quad \text{for } i = 5, 7, \dots, n. \end{aligned}$$

Obviously ℓ is a zero sum magic labeling of $E(H(t, n))$. □

Theorem 2.52. $H(t, n) \in \mathcal{V}_{a,0}$ if and only if both n and t are odd.

Proof. Proof follows from 2.50 and 2.51. □

Definition 2.15. The flower graph Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to a central vertex of the helm [6] (see figure 19).

Lemma 2.12. If $\ell : E(Fl_n) \rightarrow V_4 \setminus \{0\}$ is a labeling of Fl_n , then

$$\sum_{i=1}^n \ell^+(u_i) + \sum_{i=1}^n \ell^+(v_i) = \ell^+(w) \quad (2.27)$$

Proof. Obvious. □

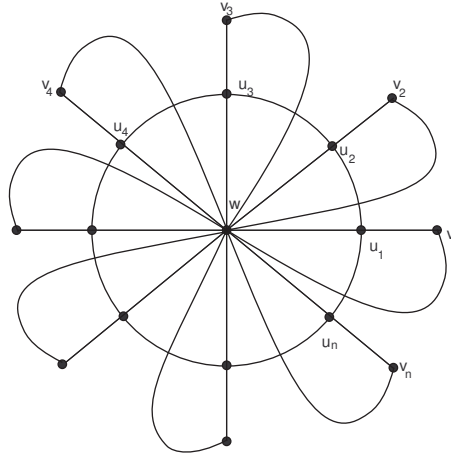


Figure 19: The flower graph: Fl_n

Theorem 2.53. $Fl_n \notin \mathcal{V}_a$ for any n .

Proof. Suppose $Fl_n \in \mathcal{V}_a$. Then by lemma 2.12, we have $na + na = a$. This implies that $a = 0$. This is a contradiction. $\square$

Theorem 2.54. $Fl_n \in \mathcal{V}_0$ for all n .

Proof. If we label all the edges by a , we obtain that, $\ell^+(u_i) = 0$, $\ell^+(v_i) = 0$ and $\ell^+(w) = 0$. $\square$

3 Conclusion

Let $V_4 = \{0, a, b, c\}$ be the Klein 4-group. In this paper, we identified a class of wheel related graphs in the following categories:

- (i) $\mathcal{V}_a$, the class of a -sum V_4 magic graphs,
- (ii) $\mathcal{V}_0$, the class of zero-sum V_4 magic graphs,
- (iii) $\mathcal{V}_{a,0}$, the class of graphs which are both a -sum and zero-sum V_4 magic.

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Competing Interests

The authors declare that no competing interests exist.

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