

1 **Original Research Article**
2 **On Parametric Generalization Of ‘Useful’**
3 **R-norm Information Measure**

4 **D.S. Hooda, Keerti Upadhyay* and D.K. Sharma**

5 Jaypee University of Engineering and Technology

6 A.B. Road, Raghogarh – 473226

7 Distt. Guna – M.P. (India).

8 ds_hooda@rediffmail.com, keerti.upadhyay13@gmail.com,

9 dilipsharmajiet@rediffmail.com

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12
13
14 ***Author details:***

15 ***D.S. Hooda, Keerti Upadhyay* and D.K. Sharma***

16
17 ***Jaypee University of Engineering and Technology***

18 ***A.B. Road, Raghogarh – 473226***

19 ***Distt. Guna – M.P. (India).***

20 ***ds_hooda@rediffmail.com, keerti.upadhyay13@gmail.com,***

21 ***dilipsharmajiet@rediffmail.com***

* Tel.: +91 7544267051(104); fax: +917544267011.

E-mail address: keerti.upadhyay13@gmail.com.

22 ABSTRACT

In the literature of information measure, there exist many well known parametric generalized information measures with their merits and limitations. In the present paper a 'useful' R-norm information measure of type α and degree β is introduced and characterized axiomatically. This new measure is parametric generalization of 'useful' R-norm information measures introduced and characterized by the authors earlier refer to Hooda et al. [9]. Properties of the new generalized 'useful' R-norm information measure of type α and degree β have also been studied. The new measure has been applied in studying the lower and upper bounds of a generalized 'useful' R-norm mean codeword length.

23

24 *Keywords:* R-norm information; Mean codeword length; Kraft's inequality; Code alphabets
25 and Holder's inequality.

26

27 1. INTRODUCTION

28

29 We consider the set of positive real numbers, not equal to 1 denoted by

30 $\mathfrak{R}^+ = \{R : R > 0 \neq 1\}$. Let Δ_n with $n \geq 2$ be the set of all probability distributions

31
$$P = \left\{ (p_1, p_2, \dots, p_n), p_i \geq 0 \text{ and } \sum_{i=1}^n p_i = 1 \right\}.$$

32 Boekee and Lubbe [4] considered R-norm information of distribution P defined for
33 $R \in \mathfrak{R}^+$ by

34
$$H_R(P) = \frac{R}{R-1} \left[1 - \left(\sum_{i=1}^n p_i^R \right)^{\frac{1}{R}} \right], R(>0) \neq 1. \quad (1.1)$$

35 The R-norm information measure (1.1) is a real function $\Delta_n \rightarrow \mathfrak{R}^+$, where $n \geq 2$ and
36 $\mathfrak{R}^+$ is the set of positive real numbers. This measure is different from entropies of
37 Shannon [13], Renyi [12], Havrda and Charvat [6], and Daroczy [5]. The most
38 interesting property is that in case $R \rightarrow 1$, R-norm information measure approaches
39 to Shannon's entropy(1948) and when $R \rightarrow \infty$, $H_R(P) \rightarrow 1 - \max p_i$,

40 $\forall i = 1, 2, \dots, n.$

41 The measure (1.1) can be generalized in so many ways; Hooda and Ram [7]
42 defined the following journalized 'useful' information measure:

43
$$H_R^\beta(P) = \frac{R}{R+\beta-2} \left[1 - \left(\sum_{i=1}^n p_i^{\frac{R}{2-\beta}} \right)^{\frac{2-\beta}{R}} \right], 0 < \beta \leq 1, R(>0) \neq 1. \quad (1.2)$$

44 The measure given by (1.2) was called generalized R-norm information measure of
 45 degree β as it reduces to (1.1) when $\beta=1$.

46 Hooda and Sharma [8] introduced and characterized parametric generalization of
 47 (1.2) as given below:

48

$$49 \quad H_R^{\alpha, \beta}(P) = \frac{R}{R + \beta - 2\alpha} \left[1 - \left(\sum_{i=1}^n p_i^{\frac{R}{2\alpha - \beta}} \right)^{\frac{2\alpha - \beta}{R}} \right], \alpha \geq 1, 0 < \beta \leq 1, R(>0) \neq 1, 0 < R + \beta \neq 2\alpha. \quad (1.3)$$

50

51
 52 (1.3) was called as the generalized R-norm information measure of type α and
 53 degree β and it reduces to (1.2) when $\alpha=1$ and further reduces to (1.1) when $\beta=1$.

54 Belis and Guiasu [2] considered qualitative aspect of events in an experiment and
 55 attached a utility distribution $U = \{(u_1, u_2, \dots, u_n), u_i > 0\}$ with the probability
 56 distribution P such that u_i is the utility of an event having probability p_i .
 57 Consequently, the following 'qualitative-quantitative' measure was defined and
 58 characterized:

$$59 \quad H(P; U) = H(p_1, p_2, \dots, p_n; u_1, u_2, \dots, u_n). \\ 60 \quad = - \sum_{i=1}^n u_i p_i \log p_i, u_i > 0, 0 < p_i \leq 1, \sum_{i=1}^n p_i = 1. \quad (1.4)$$

61 Later on the measure (1.4) was called 'useful' information by Longo [11] of the
 62 experiment.

63 To over come the limitations of the measure (1.4), Bhaker and Hooda [3] introduced
 64 and characterized the following measure of 'useful' information:

$$65 \quad H(P; U) = - \frac{\sum_{i=1}^n u_i p_i \log p_i}{\sum_{i=1}^n u_i p_i}. \quad (1.5)$$

66 Hooda et al. [9] also characterized the following 'useful' R-norm information
 67 measure:

$$H_R(P;U) = \frac{R}{R-1} \left[1 - \frac{\left(\sum_{i=1}^n u_i p_i^R \right)^{\frac{1}{R}}}{\sum_{i=1}^n u_i p_i} \right], \quad R(>0) \neq 1, \quad (1.6)$$

which reduces to Boekee and Lubbe [4] when utilities are ignored.

As the parametric generalized information measures have more potentiality and flexibility for applications point of view, so it is worthwhile to consider a two parametric generalized 'useful' R-norm information measure as given below:

$$H_R^{\alpha,\beta}(P;U) = \frac{R}{R+\beta-2\alpha} \left[1 - \frac{\left(\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}} \right)^{\frac{2\alpha-\beta}{R}}}{\sum_{i=1}^n u_i p_i} \right], \quad R(>0) \neq 1, \alpha \geq 1, 0 < \beta \leq 1, 0 < R+\beta \neq 2\alpha, \quad (1.7)$$

where U is the utility distribution corresponding to probability distribution P . We may call (1.7) as the generalized 'useful' R-norm information measure of type α and degree β .

In the present paper the 'useful' R-norm information measure type α and degree β given by (1.7) is characterized in section 2. In section 3 the properties of 'useful' R-norm information measure of type α and degree β are studied. In section 4 the new measure is applied in studying the lower and upper bounds of a generalized 'useful' R-norm mean codeword length and in the end conclusion is also given as section 5.

2. CHARACTERIZATION OF 'USEFUL' R-NORM INFORMATION MEASURE OF TYPE α AND DEGREE β

Let $S_n = \Delta_n \times \Delta_n^* \rightarrow R^+$, $n = 2, 3, \dots$ and G_n be a sequence of functions of p_i 's and u_i 's, $i = 1, 2, \dots, n$, defined over S_n satisfying the following axioms:

Axiom 2.1 $G_n(P;U) = a_1 + a_2 \sum_{i=1}^n h(p_i, u_i)$, where a_1 and a_2 are non zero constants ration numbers and

$$p, u \in J = \{(0,1) \times (0, \infty)\} \cup \{(0, y); 0 \leq y \leq 1\} \cup \{(\infty, y'); 0 \leq y' \leq \infty\}.$$

Axiom 2.2 For $P \in \Delta_n, U \in \Delta_n^*, P' \in \Delta_m$, and $U' \in \Delta_m^*$, G_{mn} satisfies the following property:

$$G_{mn}(PP':UU') = G_n(P:U) + G_m(P':U') - \frac{1}{a_1} G_n(P:U) G_m(P':U').$$

Axiom 2.3 $h(p, u)$ is a continuous function of its arguments p and u .

Axiom 2.4 Let all p_i' s are equiprobable and u_i' s are equal, then

$$G_n\left(\frac{1}{n}, \dots, \frac{1}{n}; u, \dots, u\right) = \frac{R}{R + \beta - 2\alpha} \left[1 - n^{\frac{-\beta + 2\alpha - R}{R}} \right],$$

where $n = 2, 3, \dots$, and $\alpha \geq 1$, $0 < \beta \leq 1$, $R(>0) \neq 1$, $0 < R + \beta \neq 2\alpha$.

Firstly, these lemmas are proved to facilitate the proof of the main theorem.

Lemma 2.1 Using axioms 2.1 and 2.2, we get the functional equation which is given below:

$$\sum_{i=1}^n \sum_{j=1}^m h(p_i p'_j, u_i u'_j) = \left(\frac{-a_2}{a_1} \right) \sum_{i=1}^n h(p_i, u_i) \sum_{j=1}^m h(p'_j, u'_j), \quad (2.1)$$

where $(p_i, u_i), (p'_j, u'_j) \in J$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Lemma 2.2 The continuous solution that satisfies (2.1) is the same as that of the functional equation:

$$h(pp', uu') = \left(\frac{-a_2}{a_1} \right) h(p, u) h(p', u'), \quad (2.2)$$

where a_1 and a_2 are arbitrary constants.

Proof: This Lemma can be proved on following the lines of Hooda et al [9].

Next we obtain the general continuous solution of (2.2).

Lemma 2.3 One of the general continuous solutions of equation (2.2) is given by

$$h(p, u) = \left(\frac{-a_1}{a_2} \right) \left(\frac{p^\mu u^\nu}{pu} \right)^{\frac{1}{\mu}}, \text{ where } \mu \neq 0, \nu \neq 0 \quad (2.3)$$

and

$$h(p, u) = 0. \quad (2.4)$$

Proof: Taking $g(p, u) = \left(\frac{-a_2}{a_1} \right) h(p, u)$ in (2.2), we have

$$g(pp', uu') = g(p, u)g(p', u'). \quad (2.5)$$

The most general continuous solution of (2.5) (refer to Aczel [1]) is given by

$$g(p, u) = \left(\frac{p^\mu u^\nu}{pu} \right)^{\frac{1}{\mu}}, \text{ where } \mu \neq 0 \text{ and } \nu \neq 0. \quad (2.6)$$

and

$$g(p, u) = 0. \quad (2.7)$$

On substituting $g(p, u) = \left(\frac{-a_2}{a_1} \right) h(p, u)$ in (2.6) and (2.7) we get (2.3) and (2.4)

respectively. This proves the lemma 2.3 for all rational $p \in]0, 1[$ and $u > 0$, However, by continuity it holds for all real $p \in]0, 1[$ and $u > 0$.

Theorem 2.1 Axioms 2.1 to 2.4 together with Lemmas determine the measure (1.7)

Proof: Substituting the solution (2.3) in axiom 2.1, we have

$$G_n(P; U) = a_1 \left[1 - \sum_{i=1}^n \left(\frac{p_i^\mu u_i^\nu}{p_i u_i} \right)^{\frac{1}{\mu}} \right], \quad \mu \nu \neq 0. \quad (2.8)$$

Taking $p_i = \frac{1}{n}$ and $u_i = u$ for each i in (2.8), we get

$$G_n \left(\frac{1}{n}, \dots, \frac{1}{n}, u, \dots, u \right) = a_1 \left(1 - n^{\frac{1-\mu}{\mu}} u^{\frac{\nu-1}{\mu}} \right), \quad n = 2, 3, \dots, \quad (2.9)$$

129 Axiom 2.4 together with (2.9) gives

$$130 \quad a_1 \left(1 - n^{\frac{1-\mu}{\mu}} u^{\frac{\nu-1}{\mu}} \right) = \frac{R}{R + \beta - 2\alpha} \left[1 - n^{\frac{-\beta+2\alpha-R}{R}} \right].$$

131 It implies

$$132 \quad a_1 = \frac{R}{R + \beta - 2}, \quad \mu = \frac{R}{2\alpha - \beta}, \quad \nu = 1.$$

133 Putting these values in (2.8), we have

$$134 \quad G_n(P; U) = \frac{R}{R + \beta - 2\alpha} \left[1 - \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha - \beta}}}{\sum_{i=1}^n u_i p_i} \right)^{2\alpha - \beta / R} \right]$$

$$135 \quad = H_R^{\alpha, \beta}(P; U).$$

136 Hence this completes the proof of theorem 2.1.

137 3. MAIN PROPERTIES of $H_R^{\alpha, \beta}(P; U)$

138 The following properties are satisfied by the generalized 'useful' R-norm information
139 measure $H_R^{\alpha, \beta}(P; U)$

140 **Property 3.1:** $H_R^{\alpha, \beta}(P; U)$ is a symmetric function of their arguments, if permutation
141 of p_i^s and u_i^s are taken together, i.e.

$$142 \quad H_R^{\alpha, \beta}(p_1, p_2, \dots, p_{n-1}, p_n; u_1, u_2, \dots, u_{n-1}, u_n) = H_R^{\alpha, \beta}(p_n, p_1, p_2, \dots, p_{n-1}; u_n, u_1, u_2, \dots, u_{n-1}).$$

143 .

144 **Property 3.2:** $H_R^{\alpha, \beta}\left(\frac{1}{8}, \frac{1}{8}; 1, 1\right) = 1$, in case $\alpha = \beta = 1$ and $R = 2$.

$$145 \quad \text{Proof: } H_R^{\alpha, \beta}(P; U) = \frac{R}{R + \beta - 2\alpha} \left[1 - \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{2\alpha-\beta}{R}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{R}{2\alpha-\beta}} \right]$$

146 For $i = 1, 2$, we have

$$147 \quad H_R^{\alpha, \beta}(P; U) = \frac{R}{R + \beta - 2\alpha} \left[1 - \left(\frac{u_1 p_1^{\frac{2\alpha-\beta}{R}}}{u_1 p_1} + \frac{u_2 p_2^{\frac{2\alpha-\beta}{R}}}{u_2 p_2} \right)^{\frac{R}{2\alpha-\beta}} \right].$$

148 Setting

149 $p_1 = \frac{1}{8}$, $p_2 = \frac{1}{8}$, $u_1 = 1$, $u_2 = 1$, $\alpha = 1$, $\beta = 1$ and $R = 2$, we get

$$150 \quad H_2^{1,1}\left(\frac{1}{8}, \frac{1}{8}; 1, 1\right) = 2 \left[1 - \left\{ \frac{\left(\frac{1}{8}\right)^2}{\frac{1}{8}} + \frac{\left(\frac{1}{8}\right)^2}{\frac{1}{8}} \right\}^{\frac{1}{2}} \right]$$

$$151 \quad = 2 \left[1 - \left(\frac{1}{4} \right)^{\frac{1}{2}} \right] = 1.$$

152 **Property 3.3:** Addition of one event having probability of occurrence as zero or
 153 utility as zero has no effect on 'useful' information, i.e.

$$154 \quad H_R^{\alpha, \beta}(p_1, p_2, \dots, p_n, 0; u_1, u_2, \dots, u_{n+1}) = H_R^{\alpha, \beta}(p_1, p_2, \dots, p_n; u_1, u_2, \dots, u_n) \\ = H_R^{\alpha, \beta}(p_1, p_2, \dots, p_{n+1}; u_1, u_2, \dots, u_n, 0).$$

155

156 **Proof:** Let us consider

$$\begin{aligned}
 & H_R^{\alpha, \beta}(p_1, p_2, \dots, p_n, 0; u_1, u_2, \dots, u_{n+1}) \\
 157 \quad &= \frac{R}{R + \beta - 2\alpha} \left[1 - \left\{ \frac{u_1 p_1^{\frac{R}{2\alpha - \beta}}}{u_1 p_1} + \frac{u_2 p_2^{\frac{R}{2\alpha - \beta}}}{u_2 p_2} + \dots + \frac{u_n p_n^{\frac{R}{2\alpha - \beta}}}{u_n p_n} + \dots + 0 \right\}^{\frac{2\alpha - \beta}{R}} \right] \\
 &= H_R^{\alpha, \beta}(P; U).
 \end{aligned}$$

158 **Property 3.4:** $H_R^{\alpha, \beta}(P; U)$ satisfies the non additivity of the following form:

$$159 \quad H_R^{\alpha, \beta}(P * Q; U * V) = H_R^{\alpha, \beta}(P; U) + H_R^{\alpha, \beta}(Q; V) - \frac{R + \beta - 2\alpha}{R} H_R^{\alpha, \beta}(P; U) H_R^{\alpha, \beta}(Q; V),$$

$$160 \quad \text{where } P * Q = (p_1 q_1, \dots, p_1 q_m; p_2 q_1, \dots, p_2 q_m, p_n q_1, \dots, p_n q_m),$$

$$161 \quad U * V = (u_1 v_1, \dots, u_1 v_m, u_2 v_1, \dots, u_2 v_m, u_n v_1, \dots, u_n v_m).$$

$$162 \quad \textbf{Proof:} \text{ R.H.S.} = H_R^{\alpha, \beta}(P; U) + H_R^{\alpha, \beta}(Q; V) - \frac{R + \beta - 2\alpha}{R} H_R^{\alpha, \beta}(P; U) H_R^{\alpha, \beta}(Q; V)$$

$$163 \quad = \frac{R}{R + \beta - 2\alpha} \left[1 - \left\{ \frac{\sum_{i=1}^n u_i v_j (p_i q_j)^{\frac{R}{2\alpha - \beta}}}{\sum_{i=1}^n u_i v_j p_i q_j} \right\}^{\frac{2\alpha - \beta}{R}} \right]$$

$$164 \quad = H_R^{\alpha, \beta}(P : Q; U : V)$$

$$165 \quad = \text{L.H.S.}$$

166 **Property 3.5:** Let A_i, A_j be two events having probabilities p_i, p_j and utilities

167 u_i, u_j respectively, then the utility u of compound event $A_i \cap A_j$ is defined

$$168 \quad u(A_i \cap A_j) = \frac{u_i p_i + u_j p_j}{p_i + p_j}. \quad (3.1)$$

169 **Theorem 3.1:** under the composition law (3.1) following holds:

170 $H_R^{\beta}(p_1, p_2, \dots, p_{n+1}, p', p''; u_1, u_2, \dots, u_{n+1}, u', u'')$

171 $=_n H_R^{\alpha, \beta}(p_1, p_2, \dots, p_{n+1}; u_1, u_2, \dots, u_{n+1}) + (p' + p'') H_R^{\alpha, \beta}\left(\frac{p'}{p' + p''}, \frac{p''}{p' + p''}; u', u''\right).$

172 **Proof:** $H_R^{\alpha, \beta}(p_1, p_2, \dots, p_{n+1}, p', p''; u_1, u_2, \dots, u_{n+1}, u', u'')$

173 $=_n H_R^{\alpha, \beta}(p_1, p_2, \dots, p_{n+1}; u_1, u_2, \dots, u_{n+1}) + \frac{R}{R + \beta - 2\alpha} \left[1 - \left\{ \frac{u' p'^{\frac{R}{2\alpha - \beta}}}{u' p'} + \frac{u'' p''^{\frac{R}{2\alpha - \beta}}}{u'' p''} \right\}^{\frac{2\alpha - \beta}{R}} \right]$

174

175 $=_n H_R^{\alpha, \beta} + \frac{R}{R + \beta - 2\alpha} \left[(p' + p'') - \left\{ \frac{u' \left(\frac{p'}{p' + p''} \right)^{\frac{R}{2\alpha - \beta}}}{u' \frac{p'}{p' + p''}} + \frac{u'' \left(\frac{p''}{p' + p''} \right)^{\frac{R}{2\alpha - \beta}}}{u'' \frac{p''}{p' + p''}} \right\}^{\frac{2\alpha - \beta}{R}} (p' + p'') \right]$

176

177 $=_n H_R^{\alpha, \beta} + (p' + p'') H_R^{\alpha, \beta} \left[\frac{p'}{p' + p''}, \frac{p''}{p' + p''}; u', u'' \right].$

178 This completes the proof of theorem 3.1.

179 4. APPLICATION IN SOURCE CODING

180 Here we introduce a new generalized 'useful' mean code word length as given

181 below:

182

$$L_R^{\alpha,\beta}(P;U) = \frac{R}{R+\beta-2\alpha} \left[1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right], \quad (4.1)$$

183

184

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In case $\alpha=1$ and $\beta=1$, (4.1) reduces to average code word length that was given by Hooda et al. [9] and again if utilities are ignored, i.e. $u_i=1$ for all $i=1,2,\dots,n$, it reduces to

186

$$L_R(P) = \frac{R}{R-1} \left[1 - \sum_{i=1}^n p_i D^{-l_i \left(\frac{R-1}{R} \right)} \right], \quad (4.2)$$

187

188

which is average codeword length due to Boekke and Lubbe [3]. Thus (4.1) is a valid non-additive 'useful' mean codeword length.

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190

Next we study the lower and upper bounds of $L_R^{\alpha,\beta}(P;U)$ in terms of 'useful' R-norm information measure of type α and degree β $H_R^{\alpha,\beta}(P;U)$ given by (1.7).

191

Theorem 4.1: If $l_i, i=1,2,\dots,n$ is length of codewords x_i 's, then

192

$$H_R^{\alpha,\beta}(P;U) \leq L_R^{\alpha,\beta}(P;U), R(\neq 1) > 0, \alpha \geq 1, 0 < \beta \leq 1, \quad (4.3)$$

193

under the condition

194

$$\sum_{i=1}^n u_i D^{-l_i} \leq \sum_{i=1}^n u_i p_i, \quad (4.4)$$

195

where (4.4) is the generalization of Kraft's inequality [10].

196

Proof: By Holder's inequality we have

197

$$\left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i^q \right)^{\frac{1}{q}} \leq \sum_{i=1}^n x_i y_i, \quad (4.5)$$

198 where $x_i, y_i \geq 0$ for each i and $\frac{1}{p} + \frac{1}{q} = 1$.

199 Setting $x_i = \left(\frac{u_i p_i}{\sum_{i=1}^n u_i p_i} \right)^{\frac{R}{R+\beta-2\alpha}} D^{-l_i}, y_i = \left(\frac{u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{2\alpha-\beta-R}}$

200 $p = \frac{R+\beta-2\alpha}{R}$ and $q = \frac{2\alpha-\beta-R}{2\alpha-\beta}$ and putting in (4.5), we have

201
$$\left(\frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{R}{R+\beta-2\alpha}} \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{2\alpha-\beta-R}} \leq \frac{\sum_{i=1}^n u_i D^{-l_i}}{\sum_{i=1}^n u_i p_i} \leq 1.$$

202 It implies

203
$$\left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{2\alpha-\beta-R}} \leq \left(\frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{R}{2\alpha-\beta-R}}. \quad (4.6)$$

204 **Case 1** When $R+\beta < 2\alpha$, raising Power $\frac{2\alpha-\beta-R}{R} > 0$ both sides of

205 (4.6), we have

206
$$\left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \leq \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i}.$$

207

208 Subtracting both sides from 1, we get

$$209 \quad 1 - \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \geq 1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i}. \quad (4.7)$$

210 Multiplying (4.7) by $\frac{R}{R+\beta-2\alpha} < 0$ both sides, we have

$$211 \quad \frac{R}{R+\beta-2\alpha} \left[1 - \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \right] \leq \frac{R}{R+\beta-2\alpha} \left[1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right]$$

$$212 \quad H_R^{\alpha,\beta}(P;U) \leq L_R^{\alpha,\beta}(P;U). \quad (4.8)$$

213 **Case 2** When $R+\beta > 2\alpha$, Raising power $\frac{2\alpha-\beta-R}{R} < 0$ both sides of

214 (4.6), we get

$$215 \quad \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \geq \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i}. \quad (4.9)$$

216 Subtracting both sides from 1, we get

$$217 \quad 1 - \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \leq 1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i}.$$

218

Multiplying (4.9) by $\frac{R}{R + \beta - 2\alpha} > 0$ both sides, we get

219

$$\frac{R}{R + \beta - 2\alpha} \left[1 - \frac{\left(\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}} \right)^{\frac{2\alpha-\beta}{R}}}{\sum_{i=1}^n u_i p_i} \right] \leq \frac{R}{R + \beta - 2\alpha} \left[1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right]$$

220

$$H_R^{\alpha,\beta}(P;U) \leq L_R^{\alpha,\beta}(P;U). \quad (4.10)$$

221

Thus theorem 4.1 is proved in both cases.

222

In (4.3) equality holds if and only if

223

$$D^{-l_i} = \frac{p_i^{\frac{R}{2\alpha-\beta}}}{\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i}}, \quad R > 0 (\neq 1) \quad \text{or}$$

224

$$l_i = -\log_D p_i^{\frac{R}{2\alpha-\beta}} + \log_D \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]$$

225

$$\log_D p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right] \leq l_i < \log_D p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right] + 1. \quad (4.11)$$

226

It implies

227

$$p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right] \leq D^{l_i} < D p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]. \quad (4.12)$$

228

In the next theorem, we obtain an upper bound on $L_R^{\alpha,\beta}(P;U)$ in terms of

229

$H_R^{\alpha,\beta}(P;U)$.

230

Theorem 4.2: Let $l_1, l_2, \dots, l_n$ be the codeword lengths satisfying (4.12),

231

then following inequality holds:

232

$$L_R^{\alpha,\beta}(P;U) \leq D^{\frac{2\alpha-\beta-R}{R}} H_R^{\alpha,\beta}(P;U) + \frac{R}{R+\beta-2\alpha} \left(1 - D^{\frac{2\alpha-\beta-R}{R}} \right). \quad (4.13)$$

233

Proof: From the R.H.S. of (4.12), we have

234

$$D^{l_i} < D p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]. \quad (4.14)$$

235

Here two cases arise:

236

Case 1 When $0 < R+\beta < 2\alpha$, raising both sides of (4.14) to the power

237

$\frac{2\alpha-\beta-R}{R} > 0$, we get

238

$$D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)} < D^{\frac{2\alpha-\beta-R}{R}} p_i^{\frac{R+\beta-2\alpha}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]^{\frac{2\alpha-\beta-R}{R}}. \quad (4.15)$$

239

Multiplying both sides of (4.15) by $\frac{u_i p_i}{\sum_{i=1}^n u_i p_i}$ and summing over i , we have

240

$$\frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} < D^{\frac{2\alpha-\beta-R}{R}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]^{\frac{2\alpha-\beta-R}{R}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]$$

241

or

242

$$\frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} < D^{\frac{2\alpha-\beta-R}{R}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]^{\frac{2\alpha-\beta}{R}}.$$

243

Subtracting both sides from 1 and multiplying by $\frac{R}{R+\beta-2\alpha} < 0$, we have

244

$$L_R^{\alpha,\beta}(P;U) < D^{\frac{2\alpha-\beta-R}{R}} H_R^{\alpha,\beta}(P;U) + \frac{R}{R+\beta-2\alpha} \left[1 - D^{\frac{2\alpha-\beta-R}{R}} \right]. \quad (4.16)$$

245

Similarly, we can prove that (4.16) holds when $R+\beta > 2\alpha$. Hence theorem

246

4.2 is proved

247

Thus (4.3) and (4.13) together give

248

$$H_R^{\alpha,\beta}(P:U) \leq L_R^{\alpha,\beta}(P;U) < D^{\frac{2\alpha-\beta-R}{R}} H_R^{\alpha,\beta}(P;U) + \frac{R}{R+\beta-2\alpha} \left(1 - D^{\frac{2\alpha-\beta-R}{R}} \right).$$

249

5. CONCLUSION

250

In this paper we have defined a new generalized 'useful' R-norm information

251

measure analogous to Hooda and Sharma's (2008) R-norm information measure

252

and characterized axiomatically. Some important properties have also been studied.

253 The new generalized measure has been applied in obtaining lower and upper
254 bounds of the generalized 'useful' mean code word lengths. This can be generalized
255 parametrically and applied in source coding. The measure can be further
256 generalized parametrically and can be applied in source coding.

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