

Solution of Heat and Wave – Like Equations by Adomian Decomposition Sumudu Transform Method

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Abstract

In this paper, we use Sumudu Transform Method and Adomian Decomposition Method to find solutions of heat and wave – like equations. This method is called the Adomain Decomposition Sumudu Transform Method (ADSTM).

Keywords: Adomian Decomposition Method, Sumudu Transform, heat and wave – like equations and Adomain polynomials.

1. Introduction

Heat and wave – like models are the integral part of applied mathematics and arise in various physical phenomena. Several techniques such that characteristic, modified variation iteration, Adomian decomposition method and Hé s polynomials have been used for solving these problems (see [6-10, 12-17]).

In the present paper, we introduce a new method called Adomian Decomposition Sumudu Transform Method (ADSTM) for solving the nonlinear equations. It is worth mentioning that the proposed method is an elegant combination of the Sumudu transform method and the Adomian decomposition method which was first introduced by Adomian (see [11]).

2. Sumudu Transform and Adomian Decomposition Method

Watugala (see [5]) introduced a new integral transform, named the Sumudu transform which was applied to the ordinary differential equations and partial differential equations in controlling engineering problems. Sumudu transform is defined over the set of functions;

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

$$\bar{f}(u) = S[f(t)] = \int_0^{\infty} f(ut) e^{-t} dt, u \in (-\tau_1, \tau_2)$$

By the following formula, some properties were established see ([1-3]), similarly, this transform was applied to one – dimensional neutron equation sees [4].

We consider a general nonlinear non- homogeneous partial differential equation with initial conditions of the form;

$$\begin{aligned} DU(x,t) + RU(x,t) + NU(x,t) &= g(x,t) \\ U(x,0) &= h(x), U_t(x,0) = f(x), \end{aligned} \quad (1)$$

where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is linear differential operator of less order than D ,

N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the Sumudu transform of both sides of Eq. (1), we get;

$$S[DU(x,t)] + S[RU(x,t)] + S[NU(x,t)] = S[g(x,t)]. \quad (2)$$

Using the differentiation property of the Sumudu transform and given initial conditions, we have;

$$S[U(x,t)] = u^2 S[g(x,t)] + h(x) + u f(x) - u^2 S[RU(x,t) + NU(x,t)]. \quad (3)$$

Now, apply the inverse of Sumudu transform of both sides of Eq. (3), we get;

$$U(x,t) = G(x,t) - S^{-1} [u^2 S[RU(x,t) + NU(x,t)]], \quad (4)$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, apply the Adomian decomposition method;

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t). \quad (5)$$

The nonlinear term can be decomposed as;

$$NU(x,t) = \sum_{n=0}^{\infty} A_n(U). \quad (6)$$

For some Adomian polynomials $A_n(U)$ that are given by;

$$A_n(U_0, U_1, U_2, \dots, U_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{n=0}^{\infty} \lambda^n U_n \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (7)$$

Substituting Eq. (5), and Eq. (6), in Eq. (4), we get;

$$\sum_{n=0}^{\infty} U_n(x,t) = G(x,t) - S^{-1} \left[u^2 S \left[R \sum_{n=0}^{\infty} U_n(x,t) + \sum_{n=0}^{\infty} A_n(U) \right] \right]. \quad (8)$$

So that the recursive relation is given by;

$$\begin{aligned} U_0(x,t) &= G(x,t); \\ U_{k+1}(x,t) &= -S^{-1} [u^2 S [RU_k + A_k]]. \quad k \geq 0. \end{aligned} \quad (9)$$

3. NUMERICAL APPLICATION

In this section, apply the Adomian Decomposition Sumudu Transform Method (ADSTM) to solve the heat and wave – like equations.

Example 3.1, consider the following one – dimensional initial value boundary problem;

$$U_t = \frac{1}{2} x^2 U_{xx}, \quad 0 < x < 1, t > 0, \quad (10)$$

With boundary conditions;

$$U(0,t) = 0, \quad U(1,t) = e^t, \quad (11)$$

and the initial condition;

$$U(x,0) = x^2. \quad (12)$$

Solution:

Taking the Sumudu transform of both sides of Eq. (10), and substituting the initial condition obtained;

$$S[U(x,t)] = x^2 + \frac{1}{2} x^2 u S[U_{xx}]. \quad (13)$$

Applying the inverse Sumudu transform implies that;

$$U(x, t) = x^2 + \frac{1}{2} x^2 S^{-1} [u S [U_{xx}]] , \quad (14)$$

Using the Adomian decomposition method as a series, we get;

$$\sum_{n=0}^{\infty} U_n(x, t) = x^2 + \frac{1}{2} x^2 S^{-1} \left[u S \left(\sum_{n=0}^{\infty} (U_n)_{xx}(x, t) \right) \right] ; \quad (15)$$

This leads to the recursive relation;

$$\begin{aligned} U_0(x, t) &= x^2 , \\ U_{k+1}(x, t) &= \frac{1}{2} x^2 S^{-1} [u S [(U_k)_{xx}]] . \end{aligned} \quad (16)$$

This gives;

$$\begin{aligned} U_0(x, t) &= x^2 , \\ U_1(x, t) &= \frac{1}{2} x^2 S^{-1} [u S [(U_0)_{xx}]] = x^2 t , \\ U_2(x, t) &= \frac{1}{2} x^2 S^{-1} [u S [(U_1)_{xx}]] = x^2 \frac{t^2}{2!} , \\ U_3(x, t) &= \frac{1}{2} x^2 S^{-1} [u S [(U_2)_{xx}]] = x^2 \frac{t^3}{3!} , \end{aligned} \quad (17)$$

and so on. The solution in a series form is given by;

$$U(x, t) = x^2 \left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right] . \quad (18)$$

The closed form solution is given by;

$$U(x, t) = x^2 e^t . \quad (19)$$

Example 3.2, consider the following two – dimensional initial value boundary problem;

$$U_t = \frac{1}{2} (y^2 U_{xx} + x^2 U_{yy}) , \quad 0 < x, y < 1 , t > 0 , \quad (20)$$

With boundary conditions;

$$\begin{aligned} U_x(0, y, t) &= 0 , \quad U_x(1, y, t) = 2 \sinh t , \\ U_y(x, 0, t) &= 0 , \quad U_y(x, 1, t) = 2 \cosh t , \end{aligned} \quad (21)$$

And the initial condition;

$$U(x, y, 0) = y^2 . \quad (22)$$

Solution:

In a similar way as above, we have;

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, y, t) &= y^2 + \frac{1}{2} y^2 S^{-1} \left[u S \left[\left(\sum_{n=0}^{\infty} (U_n)_{xx}(x, y, t) \right) \right] \right] \\ &+ \frac{1}{2} x^2 S^{-1} \left[u S \left[\left(\sum_{n=0}^{\infty} (U_n)_{yy}(x, y, t) \right) \right] \right] . \end{aligned} \quad (23)$$

This gives;

$$U_0(x, y, t) = y^2,$$

$$U_1(x, y, t) = \frac{1}{2} y^2 S^{-1} [u S [(U_0)_{xx}]] + \frac{1}{2} x^2 S^{-1} [u S [(U_0)_{yy}]] = x^2 t,$$

$$U_2(x, y, t) = \frac{1}{2} y^2 S^{-1} [u S [(U_1)_{xx}]] + \frac{1}{2} x^2 S^{-1} [u S [(U_1)_{yy}]] = y^2 \frac{t^2}{2!}, \quad (24)$$

$$U_3(x, y, t) = x^2 \frac{t^3}{3!},$$

$$U_4(x, y, t) = y^2 \frac{t^4}{4!},$$

and so on. The solution in a series form is given by;

$$U(x, y, t) = y^2 \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots \right) + x^2 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right). \quad (25)$$

The closed form solution is given by;

$$U(x, y, t) = y^2 \cosh t + x^2 \sinh t. \quad (26)$$

Example 3.3, let us consider the following three-dimensional inhomogeneous initial boundary value problem;

$$U_t = x^4 y^4 z^4 + \frac{1}{36} (x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}), \quad 0 < x, y, z < 1, \quad t > 0, \quad (27)$$

With boundary conditions;

$$\begin{aligned} U(0, y, z, t) &= 0, & U(1, y, z, t) &= x^4 z^4 (e^t - 1), \\ U(x, 0, z, t) &= 0, & U(x, 1, z, t) &= x^4 z^4 (e^t - 1), \\ U(x, y, 0, t) &= 0, & U(x, y, 1, t) &= x^4 y^4 (e^t - 1), \end{aligned} \quad (28)$$

and the initial condition;

$$U(x, y, z, 0) = 0. \quad (29)$$

Solution:

In a similar way as above, we have;

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, y, z, t) &= x^4 y^4 z^4 t + \frac{1}{36} x^2 S^{-1} \left[u S \left[\left(\sum_{n=0}^{\infty} (U_n)_{xx}(x, y, z, t) \right) \right] \right] \\ &+ \frac{1}{36} y^2 S^{-1} \left[u S \left[\left(\sum_{n=0}^{\infty} (U_n)_{yy}(x, y, z, t) \right) \right] \right] + \frac{1}{36} z^2 S^{-1} \left[u S \left[\left(\sum_{n=0}^{\infty} (U_n)_{zz}(x, y, z, t) \right) \right] \right]. \end{aligned} \quad (30)$$

This gives;

$$\begin{aligned} U_0(x, y, z, t) &= x^4 y^4 z^4 t, \\ U_1(x, y, z, t) &= x^4 y^4 z^4 \frac{t^2}{2!}, \\ U_2(x, y, z, t) &= x^4 y^4 z^4 \frac{t^3}{3!}, \\ U_3(x, y, z, t) &= x^4 y^4 z^4 \frac{t^4}{4!}, \end{aligned} \quad (31)$$

and so on. The solution in a series form is given by;

$$U(x, y, z, t) = x^4 y^4 z^4 \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right). \quad (32)$$

The closed form solution is given by;

$$U(x, y, z, t) = x^4 y^4 z^4 (e^t - 1). \quad (33)$$

Example 3.4, let us consider the initial – boundary value problem;

$$U_{tt} = \frac{1}{2}x^2 U_{xx}, \quad 0 < x < 1, t > 0; \quad (34)$$

with boundary conditions;

$$U(0, t) = 0, \quad U(1, t) = 1 + \sinh t, \quad (35)$$

and the initial condition;

$$U(x, 0) = x, \quad U_t(x, 0) = x^2. \quad (36)$$

Solution:

In a similar way as above, we have;

$$\sum_{n=0}^{\infty} U_n(x, t) = x + x^2 t + \frac{1}{2} x^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} (U_n)_{xx}(x, t) \right) \right] \right]. \quad (37)$$

This gives;

$$\begin{aligned} U_0(x, t) &= x^2 t + x, \\ U_1(x, t) &= x^2 \frac{t^3}{3!}, \\ U_2(x, t) &= x^2 \frac{t^5}{5!}, \\ U_3(x, t) &= x^2 \frac{t^7}{7!}, \end{aligned} \quad (38)$$

and so on. The solution in a series form is given by;

$$U(x, t) = x + x^2 \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right). \quad (39)$$

The closed form solution is given by;

$$U(x, t) = x + x^2 \sinh t. \quad (40)$$

Example 3.5, consider the following two-dimensional initial boundary value problem which describes the wave-like models;

$$U_{tt} = \frac{1}{12} (x^2 U_{xx} + y^2 U_{yy}), \quad 0 < x, y < 1, t > 0, \quad (41)$$

subject the Neumann boundary conditions;

$$\begin{aligned} U_x(0, y, t) &= 0, \quad U_x(1, y, t) = 4 \cosh t, \\ U_y(x, 0, t) &= 0, \quad U_y(x, 1, t) = 4 \sinh t, \end{aligned} \quad (42)$$

and the initial condition;

$$U(x, y, 0) = x^4, \quad U_t(x, y, 0) = y^4. \quad (43)$$

Solution:

In a similar way as above, we have;

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, y, t) &= x^4 + y^4 t + \frac{1}{12} x^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} (U_n)_{xx}(x, y, t) \right) \right] \right] \\ &\quad + \frac{1}{12} y^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} (U_n)_{yy}(x, y, t) \right) \right] \right]. \end{aligned} \quad (44)$$

The components of a solution $U(x, y, t)$ are given by;

$$\begin{aligned}
U_0(x, y, t) &= x^4 + y^4 t, \\
U_1(x, y, t) &= x^4 \frac{t^2}{2!} + y^4 \frac{t^3}{3!}, \\
U_2(x, y, t) &= x^4 \frac{t^4}{4!} + y^4 \frac{t^5}{5!}, \\
U_3(x, y, t) &= x^4 \frac{t^6}{6!} + y^4 \frac{t^7}{7!},
\end{aligned} \tag{45}$$

and so on. The solution in a series form is given by;

$$U(x, y, t) = x^4 \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \dots \right) + y^4 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right). \tag{46}$$

The closed form solution is given by;

$$U(x, y, t) = x^4 \cosh t + y^4 \sinh t. \tag{47}$$

Example 3.6, consider the initial – boundary value problem;

$$U_{tt} = x^2 + y^2 + z^2 + \frac{1}{2}(x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}), \quad 0 < x, y, z < 1, \quad t > 0, \tag{48}$$

Subject to the boundary conditions;

$$\begin{aligned}
U(0, y, z, t) &= y^2(e^t - 1) + z^2(e^{-t} - 1), \quad U(1, y, z, t) = (y^2 + 1)(e^t - 1) + z^2(e^{-t} - 1), \\
U(x, 0, z, t) &= x^2(e^t - 1) + z^2(e^{-t} - 1), \quad U(x, 1, z, t) = (x^2 + 1)(e^t - 1) + z^2(e^{-t} - 1), \\
U(x, y, 0, t) &= (x^2 + y^2)(e^t - 1), \quad U(x, y, 1, t) = (x^2 + y^2)(e^t - 1) + (e^{-t} - 1),
\end{aligned} \tag{49}$$

and having the initial conditions;

$$U(x, y, z, 0) = 0, \quad U_t(x, y, z, t) = x^2 + y^2 - z^2. \tag{50}$$

Solution:

By the same method as before, we have;

$$\begin{aligned}
\sum_{n=0}^{\infty} U_n(x, y, z, t) &= (x^2 + y^2 + z^2) \frac{t^2}{2!} + (x^2 + y^2 - z^2)t + \frac{1}{2}x^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t) \right)_{xx} \right] \right] \\
&+ \frac{1}{2}y^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t) \right)_{yy} \right] \right] + \frac{1}{2}z^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t) \right)_{zz} \right] \right].
\end{aligned} \tag{51}$$

The components of a solution $U(x, y, z, t)$ are given by;

$$\begin{aligned}
U_0(x, y, z, t) &= (x^2 + y^2 + z^2) \frac{t^2}{2!} + (x^2 + y^2 - z^2)t, \\
U_1(x, y, z, t) &= (x^2 + y^2 + z^2) \frac{t^4}{4!} + (x^2 + y^2 - z^2) \frac{t^3}{3!}, \\
U_2(x, y, z, t) &= (x^2 + y^2 + z^2) \frac{t^6}{6!} + (x^2 + y^2 - z^2) \frac{t^5}{5!}, \\
U_3(x, y, z, t) &= (x^2 + y^2 + z^2) \frac{t^8}{8!} + (x^2 + y^2 - z^2) \frac{t^7}{7!},
\end{aligned} \tag{52}$$

and so on. The solution in a series form is given by;

$$U(x, y, z, t) = (x^2 + y^2) \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right) + z^2 \left(-t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right). \tag{53}$$

The closed form solution is given by;

$$U(x, y, z, t) = (x^2 + y^2)e^t + z^2 e^{-t} - (x^2 + y^2 + z^2). \tag{54}$$

4. CONCLUSION

In the present paper, we have proposed the Adomian decomposition Sumudu transform (ADSTM) for solving heat and wave – like equations. The method gives the results reveal that the proposed method is very efficient, simple and can be applied to other applications.

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