

Original Research Article

Filtrations and Deductive Systems in BCL^+ Algebras

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Author's contribution

The only author performed the whole research work. Author YL wrote the first draft of the paper.
Author YL read and approved the final manuscript.

ABSTRACT

The aim of this paper is to introduce the two notions of filtrations and deductive systems in BCL^+ algebras, and investigate some of their related properties. Using this, we prove that the filtration is subalgebra of BCL^+ algebras; we also prove that the filtration of BCL^+ algebras is a deductive system to the filtration of BCL^+ algebras with certain relations.

Keywords: BCL algebras; BCL^+ algebras; Filtrations; Deductive systems.

Mathematics Subject Classification: 03G25, 06F99

1. INTRODUCTION

The present author (2011 and 2012) [1, 2] first introduced of BCL/BCL^+ algebras. Later on, there were more papers that the more papers have appeared in which BCL algebras and its related concepts were developed by Al-Kadi and Hosny (2013) [3] [6]; the present author (2014) [4, 5] [7] gave a characterization of a partial order and a topology in BCL^+ algebras, and discussed some distributions of BCL^+ algebras.

In this paper, we focus on how to generate a filtration by using a subset of BCL^+ algebras. We also introduce the notion of deductive systems in BCL^+ algebras, and discuss their interesting properties. Most importantly, we started using the notion of a filtration and we can to relate the deductive systems to the BCL^+ algebras.

2. PRELIMINARIES

Definition 2.1 (Definition 2.1, Liu [2]). A BCL^+ algebra is a triple $(Y; *, 1)$, where Y is a nonempty set, “ $*$ ” is a binary operation on Y , and $1 \in Y$ is an element such that the following three axioms hold for any $x, y, z \in Y$:

$$(BCL^+ 1) \quad x * x = 1.$$

$$(BCL^+ 2) \quad x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.$$

$$(BCL^+ 3) \quad ((x * y) * z) * ((x * z) * y) = (z * y) * x.$$

Definition 2.2 (Definition 2.2, Liu [2]). Suppose that $(Y; *, 1)$ is a BCL^+ algebra endowed with the ordered relation $\leq$ if and only if $x * y = 1$, for all $x, y \in Y$. Then $(Y; \leq)$ is partially ordered set and $(Y; *, 1)$ is a BCL^+ algebra corresponding to the partially ordered relation.

Note that in this ordering 1 is the largest element of Y and that Y is a bounded BCL^+ algebra (unless otherwise specified) with a smallest element 0 relative to the natural ordering.

Definition 2.3 (Definition 2.3, Liu [4]). Let $(Y; *, 1)$ be a BCL^+ algebra, for all $x, y, z \in Y$, then

$$(BCL^+ 1) \quad x \leq x.$$

$$(BCL^+ 2) \quad \text{If } x \leq y \text{ and } y \leq x \text{ then } x = y.$$

$$(BCL^+ 3) \quad ((x * y) * z) * ((x * z) * y) \leq (z * y) * x.$$

Our key results are the following theorems.

Theorem 2.1 (Theorem 2.2, Liu [2]). Assume that $(Y; *, 1)$ is a BCL^+ algebra. Then the following hold for all $x, y, z \in Y$.

$$(i) \quad (x * (x * y)) * y = 1.$$

$$(ii) \quad x * 1 = x \text{ implies } x = 1.$$

$$(iii) \quad ((x * y) * (x * z)) * (z * y) = 1.$$

$$(iv) \quad (BCL^+ 2) \quad x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.$$

Theorem 2.2 (Theorem 2.3, Liu [2]). An algebra $(Y; *, 1)$ is a BCL^+ algebra if and only if it satisfies the following conditions for all $x, y, z \in Y$.

$$(i) \quad (BCL^+ 1) \quad x * x = 1.$$

$$(ii) \quad (BCL^+ 2) \quad x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.$$

$$(iii) \quad (((x * y) * z) * ((x * z) * y)) * ((z * y) * x) = 1.$$

$$(iv) \quad x * (1 * y) = x.$$

Theorem 2.3 (Theorem 2.1, Liu [4]). Let $(Y; *, 1)$ be a BCL^+ algebra. Then the binary relation $\leq$ is a partial order on Y .

Theorem 2.4 (Theorem 2.2, Liu [4]). Suppose that $(Y; *, 1)$ is a BCL^+ algebra. We have that

$$((x * y) * (x * z)) * (z * y) = 1, \text{ for all } x, y, z \in Y,$$

where $(Y; \leq)$ is partially ordered set. Then BCL^+ algebra is $(Y; *, 1)$.

3. MAIN RESULTS

We often call the partial ordering $\leq$ defined in Definition 2.2 the BCL^+ -ordering on Y . Thus, the following basic result will prove to be useful.

Theorem 3.1. Let Y be a BCL^+ algebra. Let $x \leq y$ iff $x * y = 1$. Then $\leq$ is a partially ordering on Y ; that is $(Y; \leq)$ is a partially ordered set with greatest element 1 (and unit element) of Y .

Proof. (i) Reflexivity: Let $x \in Y$. Since $x * x = 1$. By Definition 2.2, Then we have $x \leq x$.

(ii) Anti-symmetry: Let $x, y \in Y$. Since $x \leq y$ and $y \leq x$, then $x * y = 1$ and $y * x = 1$, by Definition 2.1(BCL⁺ 2), we have $x = y$.

(iii) Transitivity: Let $x, y, z \in Y$. Since $x \leq y$ and $y \leq z$, then $x * y = 1$ and $y * z = 1$. Let $y = 1$. By Theorem 2.1(i), we have $(1 * x) * 1 = 1$, $1 * (1 * x) = 1$, by Definition 2.1(BCL⁺ 2), we have $1 * x = 1$; if $(x * 1) * x = 1$, $x * (x * 1) = 1$, by Definition 2.1(BCL⁺ 2), we have $x * 1 = x$ and $(x * 1) * x = x * (x * 1)$; if $(z * y) * 1 = 1$, $1 * (z * y) = 1$, by Definition 2.1(BCL⁺ 2), we have $z * y = 1$. Continuing this way, we see that $y * x = 1$.

Hence the transitivity of $x \leq z$ implies that

$$\begin{aligned}
x * z &= ((x * z) * 1) * (1 * 1) \\
&= ((x * z) * (y * z)) * ((x * y) * (z * y)) \\
&= ((x * y) * z) * ((x * z) * y),
\end{aligned}$$

and

$$\begin{aligned}
x * z &= ((z * x) * 1) \\
&= (z * x) * (y * x) \\
&= (z * y) * x.
\end{aligned}$$

Then we have

$$x * z = (((x * y) * z) * ((x * z) * y)) = ((z * y) * x).$$

Reasoning as above and using Theorem 2.2 (iii), we have

$$\begin{aligned}
x * z &= (((x * z) * 1) * (1 * 1)) * ((z * x) * 1) \\
&= (((x * z) * (y * z)) * ((x * y) * (z * y))) * ((z * x) * (y * x)) \\
&= (((x * y) * z) * ((x * z) * y)) * ((z * y) * x) \\
&= 1,
\end{aligned}$$

and $x \leq z$, and we know that BCL^+ algebras have partial order. Let $x \leq 1$, then $x = x * 1 = 1$, shows that 1 is the partially ordered greatest element. $\square$

Consider a bounded BCL^+ algebra $(Y; *, 1)$. We define a unary operation " $\mathfrak{R}$ " on Y by $\mathfrak{R} := x * 0$ for all $x \in Y$, and we define

$$x \vee y := (y * x) * x.$$

Note that $x \vee y$ is an upper bound of x and y .

Definition 3.1. Let Y be a BCL^+ algebra. For all $x, y, z \in Y$, we have the following identities:

- (Y1) $x * x = 1$, ~~$1 * x = x$ imply $x = 1$.~~
- (Y2) $x * (y * z) \leq (x * y) * (x * z)$.
- (Y3) $(x * y) * z \leq (x * z) * (y * z)$.
- (Y4) $x \leq y$ implies $z * x \leq z * y$ and $y * z \leq x * z$.
- (Y5) $\mathfrak{R}(0) = 1$ and $\mathfrak{R}(1) = 0$.
- (Y6) $x \leq y$ implies $\mathfrak{R}(y) \leq \mathfrak{R}(x)$.
- (Y7) $x \leq \mathfrak{R}(\mathfrak{R}(x))$.
- (Y8) $x * y \leq \mathfrak{R}(y) * \mathfrak{R}(x)$.
- (Y9) $x * \mathfrak{R}(y) = y * \mathfrak{R}(x)$.
- (Y10) ~~$((x * y) * z) * ((x * z) * y) \leq (z * y) * x$.~~ $1 * x = x$ implies $x = 1$.

Definition 3.2. Let H be a nonempty subset of a BCL^+ algebra $(Y; *, 1)$. Then we say that H is a filtration if the following three axioms are satisfied:

- (H1) $0 \in H$.
- (H2) For all $x, y \in Y$, $\mathfrak{R}(\mathfrak{R}(y) * \mathfrak{R}(x)) \in H$ and $y \in H$ imply ~~either~~ $x \in H$.
- (H3) For all $x, y, z \in Y$, $(z * (y * x)) * x \in H$.

Definition 3.3. If D be a nonempty subset of a BCL^+ algebra $(Y; *, 1)$. Then we say that D be a deductive system of Y if the following two axioms are satisfied:

- (D1) $1 \in D$.
- (D2) $x \in D$ and $x * y \in D$ imply ~~either~~ $y \in D$.

Definition 3.4. A nonempty subset S of a BCL^+ algebra $(Y; *, 1)$ is called a subalgebra $(S; *, 1)$ of BCL^+ algebra $(Y; *, 1)$ if it is a BCL^+ algebra.

Theorem 3.2. Let H be a filtration of a BCL^+ algebra $(Y; *, 1)$, and suppose $y \in H$. If $\mathfrak{R}(y) \leq \mathfrak{R}(x)$ and for all $x \in Y$, then $x \in H$.

Proof. If $\mathfrak{R}(y) \leq \mathfrak{R}(x)$, then

$$\mathfrak{R}(y) * \mathfrak{R}(x) = 1,$$

and hence

$$\mathfrak{R}(\mathfrak{R}(y) * \mathfrak{R}(x)) = \mathfrak{R}(1) = 0 \in H.$$

By Definition 3.2 (H2), we conclude that $x \in H$. $\square$

Corollary 3.1. In the situation of Theorem 3.2, if $x \leq y$ and for all $x \in Y$, then $x \in H$.

Theorem 3.3. Suppose that H is a nonempty subset of a BCL^+ algebra $(Y; *, 1)$, and let H be a filtration if and only if there exist $a_1, a_2, \dots, a_n \in H$ such that

$$a_1 * (a_2 * (\dots * (a_i * (\dots * (a_n * x) \dots)))) \in H.$$

Then $x \in H$.

Proof. Let H be a filtration, and for all $a_1, a_2, \dots, a_i, \dots, a_n \in H$, since

$$a_1 * (a_2 * (\dots * (a_i * (\dots * (a_n * x) \dots)))) \in H,$$

we have

$$a_1 \in H \text{ implies } a_2 * (\dots * (a_i * (\dots * (a_n * x) \dots))) \in H,$$

here

$$a_i \in H \text{ implies } a_{i+1} * (\dots * (a_n * x) \dots) \in H,$$

If we continue like this, we obtain in the "extreme" case that. This ratiocination proves that implies $x \in H$.

Conversely, let $a \in H$ and $a * x \in H$, then $x \in H$. $\square$

Corollary 3.2. Let H be a filtration of a BCL^+ algebra $(Y; *, 1)$, and suppose $a \in H$ for all $x \in Y$, # where $a \leq x$, then there exists $x \in H$ such that

$$(a * x) * x \in H.$$

Proof. Let $a \leq x$, by Theorem 3.3 $a * x \in H$. Since $x \in H$, then $(a * x) * x \in H$. $\square$

Theorem 3.4. Suppose that B is a subalgebra of Y . Then B is a filtration and $x \in B$ if and only if $1 * x \in B$.

Proof. If B is a filtration, then $1 * x \in B$ implies $x \in B$.

Conversely, let $x \in B$ and $x * y \in B$. Since B is subalgebra, by Definition 2.3 ($BCL^+_{\leq} 3$), we have

$$\begin{aligned} 1 * y &\leq (1 * x) * (1 * y) \\ &\leq (x * y) * x \in B. \end{aligned}$$

We see that $y \in B$, and B is a filtration. $\square$

Theorem 3.5. Suppose that $\wp$ be a nonempty family of filtrations of BCL^+ algebras $(Y; *, 1)$. Then $\bigcap \wp$ is also a filtration of BCL^+ algebras $(Y; *, 1)$.

Proof. Let $H = \bigcap \wp$, we consider

$$\wp = \{\wp_j \mid j \in J\},$$

such that $0 \in \wp_j$, for all $j \in J$, then $0 \in \wp$. Let $y * x \in \wp$ and $x \in \wp$, then for all $j \in J$, we have $y * x \in \wp_j$ and $x \in \wp_j$. Since $\wp_j$ is filtrations of Y , then $y \in \wp_j$ and, in fact, j be arbitrary, and so $y \in \wp$ also, thus $\wp$ be filtrations of Y . $\square$

Lemma 3.1. Let D be a deductive system of a BCL^+ algebra $(Y; *, 1)$, and suppose $a \leq x$ whenever $a \in D$. Then $x \in D$.

Proof. If $a \leq x$, then

$$a * x = 1 \in D.$$

By Definition 3.3 (D2), we conclude that $x \in D$. $\square$

~~If Y be commutative, then $(y * x) * x = (x * y) * y$, for all $x, y \in Y$. The key to this is the following lemma.~~

Lemma 3.2. Let Y be commutative and let for all $x, y, z \in Y$. Then the following equalities are satisfied:

$$(Y11) \quad x * (y * z) = y * (x * z).$$

$$(Y12) \quad (x * y) * z = (x * z) * y.$$

Proof. For part (Y11), suppose $x * y = y$ iff $(x * y) * y = 1$, and suppose $y * x = x$ iff $(y * x) * x = 1$. By Definition 3.1 (Y2), we see that

$$x * (y * z) \leq (x * y) * (x * z) = y * (x * z),$$

$$y * (x * z) \leq (y * x) * (y * z) = x * (y * z).$$

We obtain

$$x * (y * z) = y * (x * z).$$

For part (Y12), suppose $y * z = y$ and $z * y = z$. By Definition 3.1 (Y3), we also use the same method for $(x * y) * z = (x * z) * y$. $\square$

Lemma 3.3. Let Y be commutative and let for all $x, y \in Y$. Then the following equalities are satisfied:

$$(Y13) \quad \mathfrak{R}(\mathfrak{R}(x)) = x.$$

$$(Y14) \quad \mathfrak{R}(\mathfrak{R}(x) * \mathfrak{R}(y)) = \mathfrak{R}(y * x).$$

Proof. For part (Y13), we have

$$x \vee 0 = (0 * x) * x$$

$$= (x * 0) * 0$$

$$= \mathfrak{R}(x * 0)$$

$$= \mathfrak{R}(\mathfrak{R}(x)).$$

$$0 \vee x = (x * 0) * 0$$

$$= (0 * x) * x$$

$$= x.$$

It follows that if $x \leq (0 * x) * x$ and $x \geq (0 * x) * x$, then

$$x * ((0 * x) * x) = ((0 * x) * x) * x$$

$$= x * x$$

$$= 1.$$

By the commutativity, We get $\mathfrak{R}(\mathfrak{R}(x)) = x$.

Now, for part (Y14). By Definition 3.1 (Y6) and (Y8), we write

$$\mathfrak{R}(\mathfrak{R}(x) * \mathfrak{R}(y)) \leq \mathfrak{R}(y * x),$$

and use Lemma 3.2 (Y11) and (Y12), Definition 3.1 (Y5) and (Y1) to write

$$\begin{aligned} \mathfrak{R}(y * x) * \mathfrak{R}(\mathfrak{R}(x) * \mathfrak{R}(y)) &= ((y * x) * 0) * (((x * 0) * (y * 0)) * 0) \\ &= ((x * 0) * (y * 0)) * (((y * x) * 0) * 0) \\ &= (y * ((x * 0) * 0)) * (((y * x) * 0) * 0) \\ &= (y * ((0 * x) * x)) * ((0 * (y * x)) * (y * x)) \\ &= (y * x) * (y * x) \\ &= 1. \end{aligned}$$

In this case, we get

$$\mathfrak{R}(y * x) \leq \mathfrak{R}(\mathfrak{R}(x) * \mathfrak{R}(y)).$$

Thus $\mathfrak{R}(\mathfrak{R}(x) * \mathfrak{R}(y)) = \mathfrak{R}(y * x)$. $\square$

Lemma 3.4. Let $(Y; *, 1)$ be a BCL^+ algebra and let for all $x, y \in Y$. Then

$$(Y15) \quad x \leq (x * y) * y$$

Proof. The proof is an easy consequence of Lemma 3.2 and Definition 3.1 (Y1). $\square$

Theorem 3.6. Let H be a filtration of a BCL^+ algebra $(Y; *, 1)$, and assume that H is a nonempty subset of Y and Y is commutative. Then

$$\mathfrak{R}(x * y) \in H \text{ and } y \in H \text{ imply } x \in H \text{ for all } x, y \in Y.$$

Proof. The proof is immediate by Lemma 3.3 (Y14). $\square$

Theorem 3.7. Let H be a filtration of a BCL^+ algebra $(Y; *, 1)$. Then H is a subalgebra of Y .

Proof. Let for all $x, y, z \in H$, by Definition 2.3 ($BCL^+_{\leq}$ 3) and (Y8), and by Lemma 3.3 (Y14). We have

$$\begin{aligned} ((x * y) * z) * ((x * z) * y) &\leq (z * y) * x \\ &\leq \mathfrak{R}(x) * \mathfrak{R}(z * y) \\ &= \mathfrak{R}(x) * \mathfrak{R}(\mathfrak{R}(y) * \mathfrak{R}(z)) \in H, \end{aligned}$$

and H is filtration. By Definition 3.2 (H2), therefore, we have $\mathfrak{R}(\mathfrak{R}(y) * \mathfrak{R}(z)) \in H$, by Definition 3.2 (H3), therefore, we can write $(z * (y * x)) * x \leq \mathfrak{R}(x) * \mathfrak{R}(z * (y * x)) \in H$, we have $\mathfrak{R}(z * (y * x)) = \mathfrak{R}(\mathfrak{R}(y * x) * \mathfrak{R}(z)) \in H$, and so $y * x \in H$, we also have $x \in H$, then H is a subalgebra of Y . $\square$

Theorem 3.8. Let H be a nonempty subset of a BCL^+ algebra $(Y; *, 1)$. Then H is a filtration if and only if it is a deductive system.

Proof. Let H be a filtration and suppose $a \leq x$ with $a \in H$. Since $1 \in H$, then H satisfy Definition 3.3 (D1).

Next, we prove Definition 3.3 (D2). Let $a \in H$ and use Theorem 3.3 to write $a * x = a_1 \in H$, let for all $x \in Y$, by Corollary 3.2 we have

$$a_2 = (a * x) * x \in H.$$

This yields

$$\begin{aligned} 1 * x &= (((a * x) * x) * ((a * x) * x)) * x \\ &= (a_2 * (a_1 * x)) * x \in H, \end{aligned}$$

and $1 \in H$ imply $x \in H$. Hence H is a deductive system as desired.

Conversely, if H is a deductive system, then $1 \in H$, and we see that

$$\begin{aligned} a * (x * a) &= x * (a * a) \\ &= a * (x * x) \\ &= a * 1 \in H. \end{aligned}$$

This implies $a \in H$. Since also $x * a \in H$ for each element $x \in Y$, this the conditions Definition 3.2 (H1) and Definition 3.2 (H2) are satisfied.

Now, we prove Definition 3.2 (H3). By Lemma 3.2, Definition 3.1 (Y2) and Lemma 3.4 (Y15), then we have

$$\begin{aligned} z * (((z * (y * x)) * x)) &= (z * (y * x)) * (z * x) \\ &\geq (y * x) * x \\ &\geq y. \end{aligned}$$

Thus

$$y \leq z * (((z * (y * x)) * x)),$$

and for all $y, z \in Y$ implies $(z * (y * x)) * x \in H$. $\square$

Examples 3.1. In a BCL^+ algebra, the every order filtration is a deductive system, i.e., since $x \in D$ and $x * y \in D$ imply $y \in D$, and $z \in D$ is a lower bound of x and $x * y$, hence $z \leq y$. In fact

$$\begin{aligned} z * y &= 1 * (z * y) \\ &= (z * x) * (z * y) \\ &= z * (x * y) \\ &= x * (z * y) \\ &= 1. \end{aligned}$$

4. ON REVIEWER'S (or NAMEING) THEOREM FOR BCL^+ ALGEBRAS

In this section, we have the following theorem.

Theorem 4.1. (Reviewer, or NAMEING). If $(Y; *, 1)$ is a BCL^+ algebra, then $Y = \{1\}$.

Proof (Reviewer, or NAMEING). If we let $y = z = 1$ in $(BCL^+ 3)$, we get

$$((x * 1) * 1)((x * 1) * 1) = 1 * x,$$

so in view of $(BCL^+ 1)$,

$$(A) \quad 1 * x = 1.$$

If we let $x = 1$ in $(BCL^+ 3)$, we get, by successive applications of (A) above,

$$\begin{aligned} ((1 * y) * z) * ((1 * z) * y) &= (z * y) * 1 \\ (1 * z) * (1 * y) &= (z * y) * 1, \\ 1 * 1 &= (z * y) * 1 \end{aligned}$$

so, again in view of $(BCL^+ 1)$, we get

$$(B) \quad (z * y) * 1 = 1.$$

If, in (A) above, we set $x = z * y$, we get

$$(C) \quad 1 * (z * y) = 1.$$

Putting (B) and (C) together with $(BCL^+ 2)$, we get

$$(D) \quad z * y = 1.$$

Since z and y were general elements, we can interchange them in (D) to get

$$(E) \quad y * z = 1.$$

Then $(BCL^+ 2)$ together with (D) and (E) shows that

$$(F) \quad y = z,$$

for all elements y and z of Y . Setting $z = 1$ in (F). This completes the proof of the theorem. $\square$

Examples 4.1. Let $(Y; *, 1)$ be a BCL^+ algebra, the relation $x \leq y$ defined by $x \leq y$ if and only if $x * y = 1$ is a partial order on Y (or called the natural order on Y) with consider this order. 1 is the greatest element of Y . The set Y will be called the bounded if it has a smallest element 0 . In this case, for $x \in Y$ we denote $\mathfrak{R}(x) = x * 0$.

Examples 4.2. Consider the every join-semilattice $(Y; \vee, 1)$ with greatest element 1 ; it is possible to define the structure of a BCL^+ algebra with supremum by considering the implication $*$ defined by the order, and that is $x * y = y$ if $x \not\leq y$, and $x * y = 1$ if $x \leq y$.

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COMPETING INTERESTS

Author has declared that no competing interests exist.

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