

# Witt Groups of $\mathbb{P}^1$

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**Abstract.** In this paper we calculate the Witt groups of  $\mathbb{P}^1$ . It's a known result, but we calculate it by **another** method: we use the localisation theorem of Balmer and the excision theorem of S.Gille.

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## 1 Background

### 1.1 Witt Groups of a shifted and twisted scheme

Let  $X$  be a scheme which contains  $\frac{1}{2}$  and  $VB_X$  be the category of locally free coherent  $\mathcal{O}_X$ -modules, i.e. vector bundles. Let  $\mathcal{L}$  be a line bundle over  $X$ . We define a duality

$$\begin{aligned} * : VB_X &\longrightarrow VB_X \\ \mathcal{E} &\longmapsto *(\mathcal{E}) := \mathcal{E}^* = \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \end{aligned}$$

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which is the usual duality twisted by the line bundle  $\mathcal{L}$ . We identify naturally  $\varpi : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^{**}$ . If  $\mathcal{L} = \mathcal{O}_X$ , then  $\mathcal{E}^*$  is the usual dual and  $\varpi$  is locally given by the application that maps an element  $e$  of  $\mathcal{E}$  to the evaluation at  $e$ . The triple  $(VB_X, *, \varpi)$  is an exact category with duality.

**Definition 1.** *The Witt group of a scheme  $X$  twisted by the line bundle  $\mathcal{L}$  is:*

$$\mathcal{W}(X, \mathcal{L}) := \mathcal{W}(VB_X, *, \varpi) \quad (1)$$

For the particular case  $\mathcal{L} = \mathcal{O}_X$ , we denote  $\mathcal{W}(X, \mathcal{L}) = \mathcal{W}(X)$ .

## 1.2 Derived Witt Group

Let  $\mathcal{D}^b(VB_X)$  be the derived category of bounded complexes of vector bundles. We provide this category by a twisted shifted duality which is composed by a duality functor  $\varpi_{n, \mathcal{E}^\bullet} : \mathcal{E}^\bullet \rightarrow D_{L[n]}D_{L[n]}(\mathcal{E}^\bullet)$  and functorial isomorphisms of biduality  $D_{L[n]} : \mathcal{E}^\bullet \rightarrow \mathcal{E}^{\bullet \vee} \otimes L[n]$ .

We represent the derived Witt group by:

$$\mathcal{W}^n(X, L) := \mathcal{W}(\mathcal{D}^b(VB_X), D_{L[n]}, 1, \varpi_{n, \bullet}).$$

Elements of  $\mathcal{W}^n(X, L)$  are isometric classes of such  $(\mathcal{E}^\bullet, \phi^\bullet)$  with

$$\phi^\bullet : \mathcal{E}^\bullet \rightarrow D_{L[n]}(\mathcal{E}^\bullet)$$

is a symmetric isomorphism, with addition

$$[\mathcal{E}^\bullet, \phi^\bullet] + [\mathcal{F}^\bullet, \psi^\bullet] = [\mathcal{E}^\bullet \oplus \mathcal{F}^\bullet, \begin{pmatrix} \varphi^\bullet & 0 \\ 0 & \psi^\bullet \end{pmatrix}]$$

modulo metabolic classes, and the opposite is

$$-[\mathcal{E}^\bullet, \phi^\bullet] = [\mathcal{E}^\bullet, -\phi^\bullet].$$

Witt groups are functorial. To a morphism  $f : Y \rightarrow X$ , we have pullbacks

$$\begin{aligned} f^* : \mathcal{W}^n(X, L) &\longrightarrow \mathcal{W}^n(Y, f^*L) \\ [\mathcal{E}^\bullet, \phi^\bullet] &\longmapsto [f^*\mathcal{E}^\bullet, f^*\phi^\bullet] \end{aligned}$$

We have also a multiplication (Gille-Nenashev)

$$\begin{aligned} \mathcal{W}^n(X, L_1) \times \mathcal{W}^m(X, L_2) &\longrightarrow \mathcal{W}^{n+m}(X, L_1 \otimes L_2) \\ ([\mathcal{E}^\bullet, \phi^\bullet], [\mathcal{F}^\bullet, \psi^\bullet]) &\longmapsto [\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet, \phi^\bullet \otimes \psi^\bullet] \end{aligned}$$

This product is anticommutative:

$$[\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet, \phi^\bullet \otimes \psi^\bullet] = (-1)^{nm} [\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet, \psi^\bullet \otimes \phi^\bullet].$$

**Theorem 1.** (*Homotopic Invariance [Balmer]*)

Let  $\pi : X \times \mathbb{A}^1 \rightarrow X$  be the projection, and  $i : X \rightarrow X \times \mathbb{A}^1$  the section  $x \mapsto (x, 0)$ . Then  $\pi^*$  and  $i^*$  are inverse isomorphisms:

$$\mathcal{W}^n(X, L) \xrightleftharpoons[i^*]{\pi^*} \mathcal{W}^n(X \times \mathbb{A}^1, \pi^* L) \quad (2)$$

*Proof.* See [3]. □

To a closed subset  $Z \subset X$ , there is a subcategory  $\mathcal{D}_Z^b(VB_X) \subset \mathcal{D}^b(VB_X)$  of bounded complexes of vector bundles over  $X$  which are exact over  $U = X \setminus Z$ . The Witt groups of this subcategory are denoted  $\mathcal{W}_Z^n(X, L)$ .

**Theorem 2.** (*Localization [Balmer]*)

There is a long sequence

$$\cdots \rightarrow \mathcal{W}_Z^n(X, L) \xrightarrow{\text{Inclusion}} \mathcal{W}^n(X, L) \xrightarrow{\text{Restriction}} \mathcal{W}^n(U, L|_U) \xrightarrow{\partial} \mathcal{W}_Z^{n+1}(X, L) \rightarrow \cdots \quad (3)$$

when  $\partial$  is explicit. To a class in  $\mathcal{W}^n(U, L|_U)$ , we can write  $[\mathcal{E}_U, \phi_U]$  when  $\phi : \mathcal{E} \rightarrow D_{L[n]}(\mathcal{E})$  is a symmetric morphism of  $\mathcal{D}(VB_X)$  such that its restriction over  $U$  is an isomorphism. The mapping cone  $C(\phi)$  is exact over  $U$  and belongs to the subcategory  $\mathcal{D}_Z(VB_X)$ . Balmer provides  $C(\phi)$  with a symmetric isomorphism  $\psi : C(\phi) \rightarrow D_{L[n+1]}(C(\phi))$  which is unique up to an isometry, and we set  $\partial([\mathcal{E}, \phi]) = [C(\phi), \psi]$ .

*Proof.* See [4]. □

**Theorem 3.** (*Excision [Gille]*)

If  $i : Z \hookrightarrow X$  is the inclusion of a closed subset  $Z \subset X$  with codimension  $d$ , where  $Z$  and  $X$  are smooth, then there is a natural isomorphism

$$i_* : \mathcal{W}^n(Z, L|_Z \otimes \det N_{Z/X}) \xrightarrow{\sim} \mathcal{W}_Z^{n+d}(X, L). \quad (4)$$

*Proof.* See [14]. □

If  $i$  is the inclusion  $i : Z \hookrightarrow Z \times \mathbb{A}^d$  given by  $i(z) = (z, 0)$ , then it may be explicit. Suppose that  $x_1, x_2, \dots, x_d$  are the standard coordinates in  $\mathbb{A}^d$ , and  $K^\cdot(x_1, \dots, x_d)$  is the Koszul complex.

**Theorem 4.** Consider the inclusion  $i : Z \hookrightarrow Z \times \mathbb{A}^d$  and denote the projection  $\pi : Z \times \mathbb{A}^d \rightarrow Z$ . The isomorphism of the excision theorem is:

$$\begin{aligned} i_* : \mathcal{W}^n(Z, \mathcal{L}|_Z) &\longrightarrow \mathcal{W}_Z^{n+d}(Z \times \mathbb{A}^d, \mathcal{L}) \\ [\mathcal{E}, \phi] &\longmapsto [(\pi^* \mathcal{E}, \pi^* \phi) \otimes K^\cdot(x_1, \dots, x_d), k^\cdot] \end{aligned}$$

where  $k^\cdot$  is a symmetric isomorphism between  $[(\pi^*\mathcal{E}, \pi^*\phi) \otimes K^\cdot(x_1, \dots, x_d), k^\cdot]$  and its shifted dual.

*Proof.* See [14]. □

**Theorem 5 (Balmer).** *The Witt groups of a point  $\mathbf{Spec}(k) = \mathbb{A}_k^0 = \mathbb{P}_k^0 = * = pt$  are*

$$\mathcal{W}^n(\mathbb{A}_k^0, \mathcal{O}) = \begin{cases} \mathcal{W}(k) & \text{for } n \equiv 0 \pmod{4} \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where  $\mathcal{W}(k)$  denotes the Witt group of isometry classes of anisotropic quadratic forms over  $k$ .

*Proof.* See [5]. □

**Remark 1.** *In this work, the value of  $\mathcal{W}(k)$  is not important.*

**Theorem 6 (Walter).** *Let  $X$  be a scheme which contains  $\frac{1}{2}$ . Consider the projective space  $\mathbb{P}_X^r$  over  $X$  such that  $r \geq 1$ . Let  $m \in \mathbb{Z}/2$  and  $\mathcal{O}(m) \in \mathbf{Pic}(\mathbb{P}_X^r)/2$ .*

$$\text{If } r \text{ is even, then } \mathcal{W}^i(\mathbb{P}_X^r, \mathcal{O}(m)) = \begin{cases} \mathcal{W}^i(X) & \text{if } m \text{ is even} \\ \mathcal{W}^{i-r}(X) & \text{if } m \text{ is odd} \end{cases}$$

$$\text{If } r \text{ is odd, then } \mathcal{W}^i(\mathbb{P}_X^r, \mathcal{O}(m)) = \begin{cases} \mathcal{W}^i(X) \oplus \mathcal{W}^{i-r}(X) & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}$$

*Proof.* See [17]. □

### 1.3 Torus

Let  $\mathbb{G}_m = \mathbb{A}^1 \setminus 0$ . This  $\mathbb{G}_m$  is an affine variety:  $\mathbb{G}_m = \mathbf{Spec}(k[T, T^{-1}])$ .

**Definition 2.** *An algebraic torus is an algebraic group which is isomorphic to a finite product of  $\mathbb{G}_m$ :*

$$\mathbb{G}_m \times \mathbb{G}_m \dots \times \mathbb{G}_m = \mathbb{G}_m^n.$$

**Theorem 7.** *Let  $x$  be the coordinate on  $\mathbb{G}_m$ . For all variety  $Y$ , all line bundle  $\mathcal{L}$  over  $Y$  and all  $n$  we have the isomorphism:*

$$\begin{aligned} \mathcal{W}^n(Y, \mathcal{L}) \oplus \mathcal{W}^n(Y, \mathcal{L}) &\xrightarrow{\sim} \mathcal{W}^n(Y \times \mathbb{G}_m, \pi^*\mathcal{L}) \\ ([\mathcal{E}^\cdot, \phi^\cdot], [\mathcal{F}^\cdot, \psi^\cdot]) &\mapsto \left[ \pi^*\mathcal{E}^\cdot \oplus \pi^*\mathcal{F}^\cdot, \begin{pmatrix} \pi^*\varphi^\cdot & 0 \\ 0 & x\pi^*\psi^\cdot \end{pmatrix} \right]. \end{aligned}$$

We can denote that isomorphism by  $(1, \langle x \rangle) : (e, f) \mapsto e + \langle x \rangle f$ , when we identify every symmetric complex in  $Y$  to its pullback into  $\mathcal{W}^n(Y \times \mathbb{G}_m)$ .

**Proof.** See [14]. □

**Remark 2.** We have a long localisation exact sequence:

$$\begin{array}{ccccccc} \cdots \longrightarrow \mathcal{W}_{s_0(Y)}^n(Y \times \mathbb{A}^1, \pi^* \mathcal{L}) & \longrightarrow & \mathcal{W}^n(Y \times \mathbb{A}^1, \pi^* \mathcal{L}) & \xrightarrow{j^*} & \mathcal{W}^n(Y \times \mathbb{G}_m, \pi^* \mathcal{L}) & \xrightarrow{\partial} & \mathcal{W}_{s_0(Y)}^{n+1}(Y \times \mathbb{A}^1, \pi^* \mathcal{L}) \longrightarrow \cdots \\ & & \uparrow \pi^* \cong & \swarrow s_1^* & \nwarrow \langle x \rangle & \swarrow s_{0*} \uparrow \cong & \\ & & \mathcal{W}^n(Y, \mathcal{L}) & & & & \mathcal{W}^n(Y, \mathcal{L}) \end{array}$$

Where  $s_0 : Y \hookrightarrow Y \times \mathbb{A}^1$  is the null section and  $s_1 : Y \hookrightarrow Y \times \mathbb{A}^1$  is the constant section at 1.

**Lemma 8.** There is an isomorphism between the localisation exact sequence and the following one:

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{W}^n(Y \times \mathbb{A}^1, \pi^* \mathcal{L}) & \xrightarrow{j^*} & \mathcal{W}^n(Y \times \mathbb{G}_m, \pi^* \mathcal{L}) & \xrightarrow{\partial} & \mathcal{W}_{s_0(Y)}^{n+1}(Y \times \mathbb{A}^1, \pi^* \mathcal{L}) & \longrightarrow & 0 \\ \pi^* \uparrow \cong & & (\pi^*, \langle x \rangle \cdot \pi^*) \uparrow \cong & & s_{0*} \uparrow \cong & & \\ 0 \longrightarrow \mathcal{W}^n(Y, \mathcal{L}) & \xrightarrow{i_1} & \mathcal{W}^n(Y, \mathcal{L}) \oplus \mathcal{W}^n(Y, \mathcal{L}) & \xrightarrow{p_2} & \mathcal{W}^n(Y, \mathcal{L}) & \longrightarrow & 0 \end{array}$$

where  $i_1$  and  $p_2$  denote the inclusion of the first factor and the projection on the second one,  $s_0$  the null section and finally  $x$  is the coordinate on  $\mathbb{A}^1$  which vanishes at 0.

**Proof.** See [14]. □

**Remark 3.** The Witt groups of  $\mathbb{G}_m$  are known; if  $x_1, x_2, \dots, x_n$  are the coordinates on  $\mathbb{G}_m^n$ , then

$$\mathcal{W}^1(\mathbb{G}_m^n) = \mathcal{W}^2(\mathbb{G}_m^n) = \mathcal{W}^3(\mathbb{G}_m^n) = 0.$$

Also we have:

$$\mathcal{W}^0(\mathbb{G}_m) = \mathcal{W}(k)\langle 1 \rangle \oplus \mathcal{W}(k)\langle x \rangle,$$

and

$$\mathcal{W}^0(\mathbb{G}_m \times \mathbb{G}_m) = \mathcal{W}(k)\langle 1 \rangle \oplus \mathcal{W}(k)\langle x_1 \rangle \oplus \mathcal{W}(k)\langle x_2 \rangle \oplus \mathcal{W}(k)\langle x_1 x_2 \rangle.$$

etc.

## 2 Witt Groups of $\mathbb{P}^1$

Let  $k$  an algebraically closed field and  $\mathbb{P}^1 := \mathbb{P}_k^1$ . Let  $\mathcal{D}^b(\mathbb{P}^1) = \mathcal{D}^b(VB_{\mathbb{P}^1})$  the derived category of bounded complexes of vector bundles over  $\mathbb{P}^1$  with the usual duality  $\mathcal{E}^{\bullet\vee} = \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{E}^{\bullet}, \mathcal{O}_{\mathbb{P}^1})$ . Let calculate the Witt groups of  $\mathbb{P}^1$  using the localisation sequence with the closed subset  $Z = \{0\} \cup \{\infty\}$  and its open complementary  $\mathbb{G}_m$ . Firstly we have  $\mathbf{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$ . As Witt groups are periodic modulo 2 on  $L \in \mathbf{Pic}(X)$ , so it really remains two kinds of groups to calculate:  $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$  and  $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ .

### 2.1 Calculation of $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$

**Theorem 9.** *For all  $n \in \mathbb{N}$ ,*

$$\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \begin{cases} W(k) & \text{if } n \equiv 0 \text{ or } 1 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

*Proof.* We have the following exact sequence:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{W}^n(\mathbb{P}^1) & \longrightarrow & \mathcal{W}^n(\mathbb{G}_m) & \xrightarrow{\partial} & \mathcal{W}_{0,\infty}^{n+1}(\mathbb{P}^1) \longrightarrow \mathcal{W}^{n+1}(\mathbb{P}^1) \longrightarrow \cdots \\ & & & & \uparrow \cong & & \cong \uparrow (i_{0*}, i_{\infty*}) \\ & & & & \mathcal{W}^n(k) \oplus \mathcal{W}^n(k) & & \mathcal{W}^n(k) \oplus \mathcal{W}^n(k) \end{array}$$

As  $\mathcal{W}^n(k) = 0$  for  $n \not\equiv 0 \pmod{4}$ , we found  $\mathcal{W}^2(\mathbb{P}^1) = 0$  and  $\mathcal{W}^3(\mathbb{P}^1) = 0$ , and it becomes the exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{W}^0(\mathbb{P}^1) & \longrightarrow & \mathcal{W}^0(\mathbb{G}_m) & \xrightarrow{(\partial_0, \partial_\infty)} & \mathcal{W}_0^1(\mathbb{P}^1) \oplus \mathcal{W}_\infty^1(\mathbb{P}^1) \longrightarrow \mathcal{W}^1(\mathbb{P}^1) \longrightarrow 0 \\ & & & & \uparrow \cong & & \cong \uparrow (i_{0*}, i_{\infty*}) \\ & & & & \mathcal{W}(k) \oplus \mathcal{W}(k) & & \mathcal{W}(k) \oplus \mathcal{W}(k) \end{array}$$

We can separate two connected components 0 and  $\infty$ . Then we obtains

$$\partial_0(a\langle 1 \rangle + b\langle x \rangle) = i_{0*}(b)$$

and

$$\partial_\infty(a\langle 1 \rangle + b\langle x \rangle) = \partial_\infty(a\langle 1 \rangle + b\langle x^{-1} \rangle) = i_{\infty*}(b)$$

because  $\langle x \rangle = \langle x^{-1} \rangle$ .

Thus it grows

$$0 \rightarrow \mathcal{W}^0(\mathbb{P}^1) \rightarrow \mathcal{W}(k)\langle 1 \rangle \oplus \mathcal{W}(k)\langle x \rangle \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}} \mathcal{W}(k) \oplus \mathcal{W}(k) \rightarrow \mathcal{W}^1(\mathbb{P}^1) \rightarrow 0.$$

We define a filtration of  $\mathcal{D}^b(\mathbb{P}^1)$  as

$$0 \subseteq \mathcal{D}_{\{0,\infty\}}^b(\mathbb{P}^1) \subseteq \mathcal{D}^b(\mathbb{P}^1).$$

That gives us a short exact sequence of categories:

$$0 \rightarrow \mathcal{D}_{\{0,\infty\}}^b(\mathbb{P}^1) \hookrightarrow \mathcal{D}^b(\mathbb{P}^1) \rightarrow \mathcal{D}^b(\mathbb{P}^1)/\mathcal{D}_{\{0,\infty\}}^b(\mathbb{P}^1) \rightarrow 0.$$

Where

$$\mathcal{D}_{\{0,\infty\}}^b(\mathbb{P}^1) \cong \mathcal{D}_{\{0\}}^b(\mathbb{P}^1) \amalg \mathcal{D}_{\{\infty\}}^b(\mathbb{P}^1)$$

and

$$\mathcal{D}^b(\mathbb{P}^1)/\mathcal{D}_{\{0,\infty\}}^b(\mathbb{P}^1) \cong \mathcal{D}^b(\mathbb{P}^1 \setminus \{0, \infty\}).$$

Now

$$\mathcal{W}_{\{0,\infty\}}^p(\mathbb{P}^1) \cong \mathcal{W}_{\{0\}}^p(\mathbb{P}^1) \oplus \mathcal{W}_{\{\infty\}}^p(\mathbb{P}^1).$$

Then with respect to *the excision theorem* of **Gille**, we obtain:

$$\mathcal{W}_{\{0\}}^p(\mathbb{P}^1) := \mathcal{W}^p(\mathcal{D}_{\{0\}}^b(\mathbb{P}^1)) \cong \mathcal{W}^{p-1}(\{0\}),$$

and

$$\mathcal{W}_{\{\infty\}}^p(\mathbb{P}^1) := \mathcal{W}^p(\mathcal{D}_{\{\infty\}}^b(\mathbb{P}^1)) \cong \mathcal{W}^{p-1}(\{\infty\}).$$

Thus, if  $p \equiv 1 \pmod{4}$ , we have

$$\mathcal{W}_{\{0\}}^p(\mathbb{P}^1) \cong \mathcal{W}(k) \quad \text{et} \quad \mathcal{W}_{\{\infty\}}^p(\mathbb{P}^1) \cong \mathcal{W}(k).$$

Recall that for  $x = \frac{X_0}{X_1}$  where  $X_0 = 0$  at  $\{0\}$  and  $X_1 = 0$  at  $\{\infty\}$ , the isomorphism  $\mathcal{W}(k) \cong \mathcal{W}_{\{0\}}^p(\mathbb{P}^1)$  is described by:

$$\begin{array}{c} \langle a_1, a_2, \dots, a_r \rangle \mapsto \begin{array}{ccc} 0 & \xrightarrow{\quad} & \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r} \xrightarrow{\begin{pmatrix} X_0 & 0 & \dots & 0 \\ 0 & X_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \dots & X_0 \end{pmatrix}} & \mathcal{O}_{\mathbb{P}^1}^{\oplus r} \xrightarrow{\quad} & 0 \\ \left( \begin{array}{cccc} -a_1 X_1 & 0 & \dots & 0 \\ 0 & -a_2 X_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \dots & -a_r X_1 \end{array} \right) \downarrow & & \downarrow \left( \begin{array}{cccc} a_1 X_1 & 0 & \dots & 0 \\ 0 & a_2 X_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \dots & a_r X_1 \end{array} \right) \\ 0 & \xrightarrow{\quad} & \mathcal{O}_{\mathbb{P}^1}^{\oplus r} \xrightarrow{\begin{pmatrix} -X_0 & 0 & \dots & 0 \\ 0 & -X_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \dots & -X_0 \end{pmatrix}} & \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r} \xrightarrow{\quad} & 0 \end{array} \end{array}$$

With respect to *the localisation theorem* of **Balmer**, the spectral sequence is reduced to:

$$\dots \rightarrow \mathcal{W}^p(\mathcal{D}_{\{0,\infty\}}^b(\mathbb{P}^1)) \xrightarrow{\alpha} \mathcal{W}^p(\mathcal{D}^b(\mathbb{P}^1)) \xrightarrow{\beta} \mathcal{W}^p(\mathcal{D}^b(\mathbb{P}^1 \setminus \{0, \infty\})) \xrightarrow{\partial} \mathcal{W}^{p+1}(\mathcal{D}_{\{0,\infty\}}^b(\mathbb{P}^1)) \rightarrow \dots,$$

where  $\alpha$  is the inclusion and  $\beta$  is the restriction.

Then for  $p = 0$ , we have:

$$0 \rightarrow \mathcal{W}^0(\mathbb{P}^1) \rightarrow \mathcal{W}^0(\mathbb{G}_m) \xrightarrow{\partial} \mathcal{W}_{\{0, \infty\}}^1(\mathbb{P}^1) \rightarrow \mathcal{W}^1(\mathbb{P}^1) \rightarrow 0.$$

Recall that  $\mathbb{G}_m = \text{Spec}(k[t, t^{-1}])$  and  $\mathcal{W}^0(\mathbb{G}_m) \cong \mathcal{W}(k)\langle 1 \rangle \oplus \mathcal{W}(k)\langle x \rangle$  which is a free  $\mathcal{W}(k)$ -module of *rank* 2.

Describe now  $\partial(\langle 1 \rangle)$  and  $\partial(\langle x \rangle)$ .

•  $\langle 1 \rangle := 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0$  and  $\partial(\langle 1 \rangle) = 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \xrightarrow{1} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0$  The

$$\begin{array}{ccc} & \mathcal{O}_{\mathbb{P}^1} & \\ \downarrow 1 & & \downarrow 0 \\ 0 & \longrightarrow \mathcal{O}_{\mathbb{P}^1} & \longrightarrow 0 \end{array} \quad \begin{array}{ccc} & \mathcal{O}_{\mathbb{P}^1} & \\ \downarrow 0 & & \downarrow 0 \\ 0 & \longrightarrow \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{-1} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0 \end{array}$$

two lines of  $\partial(\langle 1 \rangle)$  are acyclic complexes so  $\partial(\langle 1 \rangle) = 0$ , then

$$\mathcal{W}(k)\langle 1 \rangle \subset \ker(\partial) = \mathcal{W}^0(\mathbb{P}^1)$$

•  $\langle x \rangle := 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0$  and  $\partial(\langle x \rangle) = 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \xrightarrow{X_0} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0$  which

$$\begin{array}{ccc} & \mathcal{O}_{\mathbb{P}^1}(-1) & \\ \downarrow X_0 & & \downarrow X_1 \\ 0 & \longrightarrow \mathcal{O}_{\mathbb{P}^1} & \longrightarrow 0 \end{array} \quad \begin{array}{ccc} & \mathcal{O}_{\mathbb{P}^1} & \\ \downarrow -X_1 & & \downarrow X_1 \\ 0 & \longrightarrow \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{-X_0} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0 \end{array}$$

prove that  $\partial(\langle x \rangle) = \langle 1 \rangle$ .

Then  $\langle 1 \rangle \mapsto (0, 0)$  and  $\langle x \rangle \mapsto (\langle 1 \rangle, \langle 1 \rangle)$ .

Next,  $\mathbb{P}^1$  with trivial duality has the following Witt groups:

$$\mathcal{W}^0(\mathbb{P}^1) = \ker(\partial) = \mathcal{W}(k)\langle 1 \rangle,$$

and

$$\mathcal{W}^1(\mathbb{P}^1) = \text{coker}(\partial) = \frac{\mathcal{W}(k) \oplus \mathcal{W}(k)}{\mathcal{W}(k)\langle \langle 1 \rangle, \langle 1 \rangle \rangle} \cong \mathcal{W}(k).$$

□

## 2.2 Calculation of $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$

**Theorem 10.** For all  $n \in \mathbb{N}$ ,  $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$ .

The groups  $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$  are more complicated. We use the theory of divisors.

**Definition 3.** An irreducible divisor on a smooth variety  $X$  is an irreducible subvariety  $Z \subset X$  of codimension 1. A divisor on a smooth variety  $X$  is a formal sum of irreducible divisors with coefficients in  $\mathbb{Z}$

$$D = a_1 Z_1 + a_2 Z_2 + \cdots + a_r Z_r.$$

Divisors on  $X$  form an abelian group  $\mathbf{Div}(X)$ . A divisor is effective if all its coefficients  $a_i \geq 0$ . We write  $D \succ E$  if  $D - E$  is effective.

For an open  $U \subset X$ , we have a restriction morphism

$$\begin{aligned} \mathbf{Div}(X) &\longrightarrow \mathbf{Div}(U) \\ D = \sum a_i Z_i &\longmapsto D|_U = \sum_{Z_i \cap U \neq \emptyset} a_i (Z_i \cap U) \end{aligned}$$

To every irreducible divisor is a non-archimedean valuation  $v_Z : K(X)^\times \rightarrow \mathbb{Z}$ , which measures the order of cancellation or the pole order of  $f \in K(X)^\times$  at the generic point of  $Z$ . The principal divisor associated to a function  $f \in K(X)^\times$  is  $\text{div}(f) = \sum_{Z \text{ irreducible}} v_Z(f) Z$ .

For each divisor  $D$  we have a subsheaf  $\mathcal{O}_X(D)$  with sections on each open set  $U \subset X$  are

$$\mathcal{O}_X(D) = \{f \in K(X)^\times / \text{div}(f)|_U \succ -D|_U\} \cup \{0\}.$$

The bundle  $\mathcal{O}_X(D)$  is the sheaf of sections of a line bundle is also noted that  $\mathcal{O}_X(D)$ . The general theorem of this theory is:

**Theorem 11.** To each smooth variety  $X$ , it corresponds an exact sequence:

$$1 \rightarrow \mathcal{O}(X)^\times \rightarrow K(X)^\times \xrightarrow{\text{div}} \mathbf{Div}(X) \xrightarrow{D \mapsto \mathcal{O}_X(D)} \mathbf{Pic}(X) \rightarrow 1.$$

let denote  $L_1 = \pi^* L \otimes_{\mathcal{O}_{Y \times \mathbb{A}^1}} \mathcal{O}_{Y \times \mathbb{A}^1}(s_0(Y))$ . It's a line bundle over  $Y \times \mathbb{A}^1$  whose sections are rational sections of  $L$  with at worst a simple pole along  $s_0(Y)$  and which are regular everywhere else.

**Lemma 12.** There is an isomorphism between the localisation exact sequence and the following one:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{W}^n(Y \times \mathbb{A}^1, L_1) & \xrightarrow{j^*} & \mathcal{W}^n(Y \times \mathbb{G}_m, L_1) & \xrightarrow{\partial} & \mathcal{W}_{s_0(Y)}^{n+1}(Y \times \mathbb{A}^1, L_1) \longrightarrow 0 \\ & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\ & & p & & (\pi^*, \langle x \rangle \cdot \pi^*) & & \sigma \\ 0 & \longrightarrow & \mathcal{W}^n(Y, L) & \xrightarrow{i_2} & \mathcal{W}^n(Y, L) \oplus \mathcal{W}^n(Y, L) & \xrightarrow{p_1} & \mathcal{W}^n(Y, L) \longrightarrow 0 \end{array}$$

where  $i_1$  and  $p_2$  denote the inclusion of the first factor and the projection on the second one,  $s_0$  the null section and finally  $x$  is the coordinate on  $\mathbb{A}^1$  which vanishes at 0.

Note that the isomorphism in middle of diagrams of this lemma and the lemma is the same  $\pi^*L$  and  $L_1$  have the same restrictions to  $Y \times \mathbb{G}_m$ , but the role of factors of the direct sum in the bottom exact sequence is reversed.

**Lemma 13.** *Let  $\xi : L \xrightarrow{\cong} L_1$  be an isomorphism of line bundles over a variety  $X$ . Then*

$$\begin{aligned} \xi_{\sharp} : \mathcal{W}^n(X, L) &\longrightarrow \mathcal{W}^n(X, L_1) \\ [\mathcal{E}^{\cdot}, \phi^{\cdot}] &\longmapsto [\mathcal{E}^{\cdot}, (1_{\mathcal{E}^{\cdot \vee [n]}} \otimes \xi) \circ \phi^{\cdot}] \end{aligned}$$

is an isomorphism between derived Witt groups which is compatible with restriction to open subsets and to localisation long exact sequences.

We identify  $\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(0)$ . But  $\mathbb{P}^1$  is the union of two open subsets  $\mathbb{A}_0^1 = \mathbf{Spec}(K[x])$  and  $\mathbb{A}_{\infty}^1 = \mathbf{Spec}(K[x^{-1}])$ . We have  $\mathcal{O}_X(0)(k[x]) = x^{-1}k[x]$  and  $\mathcal{O}_X(0)(k[x^{-1}]) = xk[x^{-1}]$ .

*Proof of theorem 10.* For  $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ , we identify  $\mathcal{O}_{\mathbb{P}^1}(1) \cong t^{-1} \cdot \mathcal{O}_{\mathbb{P}^1} = L(0)$ , all germs of rational functions with at worst a simple pole at 0 and regular elsewhere. Then the localisation sequence becomes:

$$0 \longrightarrow \mathcal{W}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \longrightarrow \mathcal{W}^0(\mathbb{G}_m) \xrightarrow{(\beta_0, \beta_{\infty})} \mathcal{W}_0^0(\mathbb{P}^1, t^{-1} \cdot \mathcal{O}_{\mathbb{P}^1}) \oplus \mathcal{W}_{\infty}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{\partial} \mathcal{W}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \longrightarrow 0.$$

Here we have  $(\beta_0, \beta_{\infty}) : \mathcal{W}^0(\mathbb{G}_m) \rightarrow \mathcal{W}(k) \oplus \mathcal{W}(k)$ , but  $\mathcal{W}^0(\mathbb{G}_m) \cong \mathcal{W}(k)\langle 1 \rangle \oplus \mathcal{W}(k)\langle t \rangle$ . Thus  $\beta_0 : a\langle 1 \rangle + b\langle t \rangle \mapsto a$   $\beta_{\infty} : a\langle 1 \rangle + b\langle t \rangle \mapsto b$ . Then  $(\beta_0, \beta_{\infty})$  is an isomorphism and its kernel is  $\mathbf{ker}(\beta_0, \beta_{\infty}) = \mathcal{W}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$ , and its cokernel is  $\mathbf{coker}(\beta_0, \beta_{\infty}) = \mathcal{W}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$ .  $\square$

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