Galerkin RBF for Integro-Differential Equations

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Abstract

Two methods based on the Galerkin method with Radial basis Functions (RBF) as bases are applied to solve integro-differential equations (IDEs). In the first approach, direct Galerkin RBF method, the unknown function of the IDE is approximated by RBFs and then the derivatives of it are replaced by the derivatives of RBFs. In the second one, indirect Galerkin RBF method, the derivative of the unknown function is approximated by RBFs and then lower order derivatives and unknown function itself are computed by integrating of RBFs. Therefore the Galerkin method is applied to compute these coefficients. Double integrals that appeared in the process, can be reduced to single integrals by using a formula of iterated integrals. In complicated cases, single integrals approximated by Legendre-Gauss-Lobatto quadrature. Illustrative examples are included to demonstrate the validity and applicability of the presented techniques. ~~The results of applying the proposed method are compared with those of the direct Galerkin method.~~ A comparison of applying these methods shows the efficiency and high accuracy of the indirect Galerkin RBF method rather than direct Galerkin RBF method.

Keywords: Integro-Differential Equations (IDEs); Radial Basis Functions (RBF); Galerkin Method; Legendre-Gauss-Lobatto quadrature; Iterated integrals

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1 Introduction

Integral and integro-differential equations arise from various applications, like physics, engineering, biology, medicine, economics, potential theory and many others (See [Wazwaz (2011); Delves and

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Mohamed (1985); Kress (1989)] and references therein). In recent years, a lot of attention has been devoted to the study of integro-differential equations (IDEs) such as sine-cosine wavelets [Kajani et al. (2006)], Bessel functions [Parandin et al. (2013)] Homotopy Analysis Method [Fariborzi Araghi and Behzadi (2011)], Homotopy Perturbation Method [(Agadjanov)], Adomian Decomposition Method [Biazar (2005); Biazar et al. (2003)], Variational iteration method [Biazar et al. (2010); Wang and He (2007)], differential transform method[Arikoglu and Ozkol (2008)], Compact Finite Difference Method [Zhao and Corless (2006)], and Galerkin method [Turkyilmazoglu (2014a,b)].
The aim of this paper is to approximate the solution of integro-differential equations of the second kind, in the following general form

$$y'(x) = f(x) + \int_a^b K(x,t)y(t)\,dt, \qquad a \le x \le b, \tag{1.1}$$

$$y(a) = \alpha_0, \tag{1.2}$$

where $y(x)$ is an unknown real function defined on the closed interval $[a,b]$, and $f(x)$ is an analytic known function. Moreover the kernel $K(x,t)$ is defined on the interval $a \le x, t \le b$.

Since 1990, radial basis function method [Hardy (1971)] is used as a well-known family of meshless methods to approximate the solutions of various types of linear and nonlinear functional equations such as Partial Differential Equations (PDEs), Ordinary Differential Equations (ODEs), Integral Equations (IEs), and Integro-Differential Equation (IDEs) [Hardy (1971); Kansa (1990); Platte and Driscoll (2004); Larsson and Fornberg (2003); Dehghan and Shokri (2007, 2008); Franke and Schaback (1997)]. The most of these methods are used radial functions as bases for collocation method. Meshless Galerkin methods using radial basis functions were first introduced by H. Wendland [Wendland (1999)]and then a few researchers used the method for solving PDEs [Chernih and Le Gia (2013); Kormann and Larsson (2013); Mai-Duy and Tran-Cong (2009)]. In recent years, there are some applications of Galerkin method [Turkyilmazoglu (2013, 2014a,b)].

In this paper, RBFs are used in the Galerkin method as basis functions in an indirect approach. Indirect Galerkin method approximates the derivative of unknown function of the equation by a finite linear combination of basis functions.

This paper is organized as follows. In Section 2, the radial basis functions are introduced. Section 3, reviews the Legendre-Gauss-Lobatto integration method. Section 4, as the main part, presents the solution of integro-differential equations by direct and indirect Galerkin RBF methods. Numerical examples are presented in Section 5. A conclusion is drawn in the Section 6.

2 Radial Basis Functions

Approximation of a function $u : \mathbb{R}^d \to \mathbb{R}$ by RBF can be presented as the following [Buhmann (2003)]

$$s_N(\mathbf{x}) := \sum_{i=0}^{N} \lambda_i \phi\left(\|\mathbf{x} - \mathbf{x}_i\|\right), \quad \mathbf{x} \in \mathbb{R}^d. \tag{2.1}$$

Where $\phi : [0,\infty) \to \mathbb{R}$ is a fixed univariate function, the coefficients $(\lambda_i)_{i=0}^N$ are real numbers, $(\mathbf{x}_i)_{i=0}^N$ is a set of interpolation points in $\mathbb{R}^d$, and $\|\cdot\|$ is the Euclidean norm.
Eq. (2.1) can be written as follows

$$s_N(\mathbf{x}) := \sum_{i=0}^{N} \lambda_i \phi_i(\mathbf{x}) = \mathbf{\Phi}^T(\mathbf{x})\mathbf{\Lambda}, \tag{2.2}$$

where

$$\phi_i(\mathbf{x}) = \phi(\|\mathbf{x} - \mathbf{x}_i\|),$$
$$\mathbf{\Phi}(\mathbf{x}) = [\phi_0(\mathbf{x}), \phi_1(\mathbf{x}), \ldots, \phi_N(\mathbf{x})]^T,$$
$$\mathbf{\Lambda} = [\lambda_0, \lambda_1, \ldots, \lambda_N]^T.$$

Consider $N+1$ distinct support points $(\mathbf{x}_j, u(\mathbf{x}_j)),\ j = 0, 1, \ldots, N$. One can use interpolation conditions to find λ_is by solving the following linear system

$$\mathbf{A}\mathbf{\Lambda} = \mathbf{u},$$

in which

$$\mathbf{A} = [\phi(\|\mathbf{x}_j - \mathbf{x}_i\|)]_{i,j=0}^N,$$

and $\mathbf{u} = [u(\mathbf{x}_0), u(\mathbf{x}_1), \ldots, u(\mathbf{x}_N)]^T$.

Some well-known RBFs are listed in Table 1, where the Euclidian distance r is real and non-negative, and c is a positive scalar, called the shape parameter.

Table 1: Some well-known RBFs

Name of the RBF	Definition
Gaussian	$\phi(r) = e^{-(cr)^2}$
Inverse Quadric	$\phi(r) = \frac{1}{r^2+c^2}$
Hardy Multiquadric	$\phi(r) = \sqrt{r^2+c^2}$
Inverse Multiquadric	$\phi(r) = \frac{1}{\sqrt{r^2+c^2}}$
Cubic	$\phi(r) = r^3$
Thin Plate Spline	$\phi(r) = r^2 \log(r)$
Hyperbolic Secant	$\phi(r) = \mathrm{sech}(\frac{r}{c})$

Some of RBFs are unconditionally positive definite (e.g. Gaussian or Inverse Multiquadrics) to guarantee that the resulting system is solvable, and some of them are conditionally positive definite. Although, some of RBFs are conditionally positive definite functions, polynomials are augmented to Eq. (2.1) to guarantee that the outcome interpolation matrix is invertible. Such an approximation can be expressed as follows

$$s(\mathbf{x}) = \sum_{i=0}^{N} \lambda_i \phi(\|\mathbf{x} - \mathbf{x}_i\|) + \sum_{i=1}^{l} \lambda_{N+i} p_i(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \tag{2.3}$$

where $p_i(\mathbf{x}),\ i = 1, \ldots, l$, are polynomials on $\mathbb{R}^d$ of degree at most m, and $l = \binom{m-1}{d}$. Here l is the dimension of the linear space Π_{m-1}^d of polynomials of total degree less than or equal to $m-1$, of d variables.

Collocation method is used to determinate the coefficients $(\lambda_0, \lambda_1, \ldots, \lambda_N)$ and $(\lambda_{N+1}, \lambda_{N+2}, \ldots, \lambda_{N+l})$. This will produce $N+1$ equations at $N+1$ points. l additional equations is usually written in the following form

$$\sum_{i=0}^{N} \lambda_i p_j(\mathbf{x}_i) = 0, \quad j = 1, \ldots, l. \tag{2.4}$$

3 Legendre-Gauss-Lobatto quadrature

Let L_N be the well-known Legendre polynomial of order N, on the interval $[-1,1]$. Then the Legendre-Gauss-Lobatto nodes are

$$\begin{aligned}(1-x_j^2)L_N'(x_j)=0,\\ -1=x_0<x_1<\ldots<x_N=1,\end{aligned} \tag{3.1}$$

where $x_m, 1\leq m\leq N-1$ are the zeros of L_N', where L_N' is the derivative of L_N with respect to $x\in[-1,1]$. No explicit formula for the nodes (3.1) is known, and so they are computed numerically using sub-routines [Canuto et al. (1988); Elnagar and Kazemi (1998); Elnagar and Razzaghi (1997)]. Now we approximate the integral of f on $[-1,1]$ as

$$\int_{-1}^{1} f(x)\,dx=\sum_{i=0}^{N} w_j f(x_j), \tag{3.2}$$

where x_j in Eq. (3.1) are Legendre-Gauss-Lobatto nodes and w_j are the weights given in [Shen and Tang (2005)]

$$w_j=\frac{2}{N(N+1)}\frac{1}{(L_N(x_j))^2},\quad j=0,1,\ldots,N. \tag{3.3}$$

Note that the integration in Eq.(3.2) is exact whenever $f(x)$ is a polynomial of degree $\leq 2N+1$.

4 Galerkin RBF Method

For implementing the Galerkin method, our functions belong to a complete normed linear vector space, with the following inner product

$$<f,g>:=\int_a^b f(x)g(x)dx, \tag{4.1}$$

and the following norm

$$\|f\|=\left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}}. \tag{4.2}$$

4.1 Galerkin RBF Method

In the direct method, the solution of Eq. (1.1) approximated by a finite linear combination of radial basis functions, as follows

$$y(x)=\sum_{i=0}^{N}\lambda_i\phi_i(x)+\sum_{i=1}^{m-1}\lambda_{N+i}p_i(x) \tag{4.3}$$

where

$$\phi_i(x)=\phi(\|x-x_i\|),$$

x_i, $i=0,1,...,N$, are centers, and $p_i(\mathbf{x}),\ i=1,\ldots,m-1$, are polynomials on $\mathbb{R}$ of degree at most $m-1$. By defining $\phi_{N+i}(x)=p_i(x)$, for $i=1,\ldots,m-1$, we can write Eq. (4.3) in the following form

$$y(x)=\mathbf{\Phi}^T(x)\mathbf{\Lambda}, \tag{4.4}$$

where

$$\mathbf{\Phi}^T(x)=[\phi_0(x),\phi_1(x),\ldots,\phi_{N+m-1}(x)],$$

and

$$\mathbf{\Lambda} = [\lambda_0, \lambda_1, \ldots, \lambda_{N+m-1}]^T$$

is an unknown vector.
By differentiating from Eq. (4.3), we obtain

$$y'(x) = \mathcal{D}\mathbf{\Phi}^T(x)\mathbf{\Lambda} \tag{4.5}$$

where

$$\mathcal{D}\mathbf{\Phi}^T(x) = [\phi_0'(x), \phi_1'(x), \ldots, \phi_{N+m-1}'(x)]$$

Substituting Eqs. (4.3) and (4.5) in Eq. (1.1), leads to

$$\mathcal{D}\mathbf{\Phi}^T(x)\mathbf{\Lambda} = f(x) + \int_a^b K(x,t)\mathbf{\Phi}^T(x)\mathbf{\Lambda}dt. \tag{4.6}$$

Inner product of two sides of Eq. (4.6) by $\phi_j(x)$, $j = 0, 1, \ldots, N+m-1$, generates a system of linear equations for the unknowns λ_i, $i = 0, \ldots, N+m-1$.

$$< \phi_j(x), \mathcal{D}\left(\mathbf{\Phi}^T(x)\right) > \mathbf{\Lambda} =< \phi_j(x), f(x) > + < \phi_j(x), \int_a^b K(x,t)\left(\mathbf{\Phi}^T(t)\right)dt > \mathbf{\Lambda}. \tag{4.7}$$

where $j = 0, 1, \ldots, N+m-1$. By using the Legendre-Gauss-Lobatto quadrature, we can approximate the integrals appeared in Eq. (4.7).
One can substitute initial condition

$$\mathbf{\Phi}^T(a)\mathbf{\Lambda} = \alpha_0,$$

for an equation in the foregoing linear system.

4.2 Indirect Galerkin RBF Method

In order to apply indirect Galerkin RBF (IGRBF) method, let's approximate $y'(x)$ in terms of radial basis functions $\phi_i(x)$, $i = 0, \ldots, N$ and polynomials $p_i(x) =: \phi_{N+i}(x)$, $i = 1, \ldots, m-1$ as follows

$$\begin{aligned} y'(x) &= \sum_{i=0}^{N} \lambda_i \phi_i(x) + \sum_{i=1}^{m-1} \lambda_{N+i} p_i(x) \\ &= \mathbf{\Phi}^T(x)\mathbf{\Lambda} \end{aligned} \tag{4.8}$$

Integrating of the Eq. (4.8) yields an expression for the original function:

$$\begin{aligned} \int_a^x y'(t)dt &= y(x) - y(a) \\ &= \sum_{i=0}^{N+m-1} \lambda_i \int_a^x \phi_i(t)\,dt = \mathcal{I}\mathbf{\Phi}^T(x)\mathbf{\Lambda} \end{aligned}$$

or

$$y(x) = \mathcal{I}\mathbf{\Phi}^T(x)\mathbf{\Lambda} + y(a), \tag{4.9}$$

where $y(a)$ is known from the initial condition and

$$\mathcal{I}\mathbf{\Phi}^T(x) = \left[\int_a^x \phi_0(t)\,dt, \int_a^x \phi_1(t)\,dt, \ldots, \int_a^x \phi_{N+m-1}(t)\,dt\right].$$

Substituting from Eqs. (4.8) and (4.9) in Eq. (1.1), leads to

$$\mathbf{\Phi}^T(x)\mathbf{\Lambda} = f(x) + \int_a^b K(x,t)\,\mathcal{I}\mathbf{\Phi}^T(t)\mathbf{\Lambda}dt,$$

or

$$\left(\mathbf{\Phi}^T(x) - \int_a^b K(x,t)\mathcal{I}\mathbf{\Phi}^T(t)dt\right)\mathbf{\Lambda} = f(x). \tag{4.10}$$

Inner product of two sides of Eq. (4.10) by $\phi_j(x)$, $j = 0, 1, \ldots, N+m-1$, generates a system of linear equations for the unknowns λ_i, $i = 0, \ldots, N+m-1$

$$< \phi_j, \left(\mathbf{\Phi}^T(x) - \int_a^b K(x,t)\mathcal{I}\mathbf{\Phi}^T(t)dt\right) > \mathbf{\Lambda} =< \phi_j, f(x) > . \tag{4.11}$$

Now numerical quadratures can be applied. Finally approximate solution of Eq. (1.1), is given by (4.9).

The convergency of radial basis function interpolation has been discussed by Buhmann [Buhmann (2003, 1989)] and other researchers [Jackson (1988); Schaback (1996); Wendland (1998)]. Also the error analysis of the IGRBF method are discussed in [Mai-Duy and Tran-Cong (2009); Wendland (1999)].

5 Numerical Examples

In this section, two examples are provided to illustrate the efficiency of this approach. For the sake of comparing purposes, we use the norm two of errors, and centers, x_i are roots of Legendre polynomial of degree N.

5.1 Example

Let us consider the following first order Volterra IDE

$$y' = 1 - 2x\sin(x) + \int_0^x y(t)dt, \quad y(0) = 0, \quad 0 \le x \le 1. \tag{5.1}$$

with the exact solution $y(x) = x\cos(x)$.
Errors of the numerical solutions of Eq. (5.1) by Galerkin RBF (GRBF) and indirect Galerkin RBF (IGRBF) for $N = 10, 15, 20$, and three different RBFs, Gaussian (GA), Multiquadric (MQ), and Inverse Multiquadric (IMQ), are shown in Table 2

Table 2: Errors for Example 1

	GA		MQ		IMQ	
N	GRBF	IGRBF	GRBF	IGRBF	GRBF	IGRBF
10	2.8852e-07	4.1743e-08	3.2830e-07	3.4020e-08	8.0619e-07	8.9479e-08
15	2.6959e-07	1.9768e-07	1.9996e-06	8.0225e-07	3.6005e-07	6.0615e-08
20	9.0134e-07	3.0391e-07	4.7677e-07	1.0683e-07	1.1740e-06	2.5984e-07

5.2 Example

Consider the following IDE

$$y'(x) = (1+x)\cos(x) - \sin(x) + \int_0^x ty(t)dt, \; y(0) = 0, \;\; 0 \le x \le 1. \tag{5.2}$$

The exact solution is $y(x) = \sin(x)$.
Errors of the numerical solutions of Eq. (5.2) by GRBF and IGRBF for $N = 10, 15, 20$, and three different RBFs, Gaussian (GA), Multiquadric (MQ), and Inverse Multiquadric (IMQ), are shown in Table 3.

Table 3: Errors for Example 2

	GA		MQ		IMQ	
N	GRBF	IGRBF	GRBF	IGRBF	GRBF	IGRBF
10	2.4904e-07	1.2027e-08	8.2300e-08	1.3740e-08	3.4477e-07	2.7402e-08
15	1.6148e-07	9.7282e-07	1.9535e-07	9.2351e-08	2.2556e-07	3.0249e-08
20	5.6848e-06	3.4122e-07	2.3654e-07	3.9182e-08	3.2451e-07	3.1201e-08

6 Conclusion

Two new approaches based on Galerkin and RBF method are considered for solving integro-differential equations. The first one, used RBFs as base functions for Galerkin method by approximating the unknown function of Eq. (1.1). In the second one, indirect Galerkin RBF method, Galerkin method used RBFs as base functions by approximating the derivative of the unknown function in Eq. (1.1). Two numerical examples are presented to demonstrate that these methods are very effective and useful for finding approximate solutions of integro-differential equations. The results show that IGRBF method is more effective than GRBF method.

References

Yusufolu (Agadjanov), E. (2009). Improved homotopy perturbation method for solving fredholm type integro-differential equations. *Chaos, Soliton. Fract*, 41(1):28-37.

Arikoglu, A. and Ozkol, I. (2008). Solutions of integral and integro-differential equation systems by using differential transform method. *Comput. Math. Appl.*, 56(9):2411-2417.

Biazar. J. (2005). Solution of systems of integraldifferential equations by adomian decomposition method. *Appl. Math. Comput.*, 168(2):1232-1238.

Biazar, J. Babolian, E., and Islam, R. (2003). Solution of a system of volterra integral equations of the first kind by adomian method. *Appl. Math. Comput.*, 139(23):249-258.

Biazar, J., Gholami Porshokouhi, M., and Ebrahimi, H. (2010). Variational iteration method for linear and nonlinear systems of integro-differential equations. *Int. Math. Forum*, 5(61-54):3185-3191.

Buhmann, M.D. (2003). *Radial basis functions: theory and implementations*. Cambridge University Press.

Buhmann, M.D. (1989). *Multivariable interpolation using radial basis functions*. PhD thesis, University of Cambridge.

Canuto, C., Hussaini, M.Y., Quarteroni, A., and Zang, T.A. (1988). *Spectral methods in fluid dynamics*. Springer series in computational physics. Springer-Verlag.

Chernih, A. and Le Gia, Q. T. (2013) Multiscale methods with compactly supported radial basis functions for galerkin approximation of elliptic pdes. *IMA J. Numer. Anal.*, 2013. doi: 10.1093/imanum/drt004.

Dehghan, M. and Shokri, A. (2007). A numerical method for two-dimensional schrdinger equation using collocation and radial basis functions. *Comput. Math. Appl.*, 54(1):136-146.

Dehghan, M. and Shokri, A. (2008) A numerical method for solution of the two-dimensional sine-gordon equation using the radial basis functions. *Math. Comput. Simul.*, 79(3):700-715.

Delves, L.M. and Mohamed, J.L. (1985). *Computational Methods for Integral Equations*. Cambridge University Press.

Elnagar, G.N. and Kazemi, M.A. (1998). Pseudospectral legendre-based optimal computation of nonlinear constrained variational problems. *J. Comput. Appl. Math.*, 88(2):363-375.

Elnagar, G.N. and Razzaghi, M. (1997). Short communication: A collocation-type method for linear quadratic optimal control problems. *Optim. Control. Appl. Meth.*, 18(3):227-235.

Fariborzi Araghi, M.A. and Sadigh Behzadi, S. (2011). Numerical solution of nonlinear volterra-fredholm integro-differential equations using homotopy analysis method. *J. Appl. Math. Comput.*, 37(1-2):1-12.

Franke, C. and Schaback, R. (1997). Solving partial differential equations by collocation using radial basis functions. Appl. Math. Comput., 93, 73-82.

Hardy, R.L. (1971). Multiquadric equations of topography and other irregular surface. J. Geophys. Res., 76, 1905-1915.

Jackson, I.R.H. (1988). Radial basis function methods for multivariable approximation. PhD dissertation, University of Cambridge.

Tavassoli Kajani, M., Ghasemi, M., and Babolian, E. (2006). Numerical solution of linear integro-differential equation by using sinecosine wavelets. *Appl. Math. Comput.*, 180(2):569-574.

Kansa, E.J. (1990). Multiquadrics scattered data approximation scheme with applications to computational fluid dynamics II. Solutions to hyperbolic, parabolic, and elliptic partial differential equations. Comput. Math. Appl., 19, 147-161.

Kormann, K. and Larsson, E. (2013). A galerkin radial basis function method for the schrödinger equation. *SIAM J. Sci. Comput.*, 35(6):A2832-A2855.

Kress, R. (1989). Linear Integral Equations. Springer-Verlag, Berlin.

Larsson, E. and Fornberg, B. (2003). A numerical study of some radial basis function based solution methods for elliptic PDEs. Comput. Math. Appl., 46, 891-902.

Mai-Duy, N. and Tran-Cong, T. (2009) An integrated-rbf technique based on galerkin formulation for elliptic differential equations. *Eng. Anal. Bound. Elem.*, 33(2):191-199.

Parandin, N., Chenari, S., and Heidari, S. (2013). A numerical method for solving linear fredholm integrodifferential equations of the first order. *J. Basic. Appl. Sci. Res.*, 3(1s):192-195.

Platte, R.B. and Driscoll, T.A. (2004). Computing eigenmodes of elliptic operators using radial basis functions. *Comput. Math. Appl.*

Schaback, R. (1996). Approximation by radial basis functions with finitely many centers. *Constr. Approx.*, 12(3):331-340.

Shen, J. and Tang, T. (2005). *High Order Numerical Methods and Algorithms*. Chinese Science Press.

Turkyilmazoglu, M. (2014). An effective approach for numerical solutions of high-order fredholm integro-differential equations. *Appl. Math. Comput.*, 227 (0): 384-398.

Turkyilmazoglu, M. (2014). High-order nonlinear volterra-fredholm-hammerstein integro-differential equations and their effective computation. *Appl. Math. Comput.*, 247:410-416.

Turkyilmazoglu, M. (2013). Effective computation of exact and analytic approximate solutions to singular nonlinear equations of lane-emden-fowler type. *Appl. Math. Model.*, 37(14):7539-7548.

Wang, S.-Q. and He, J.H. (2007). Variational iteration method for solving integro-differential equations. *Phys. Lett. A*, 367(3):188-191.

Wazwaz, A.-M. (2011). *Linear and Nonlinear Integral Equations, Methods and Applications*. Springer-Verlag, Berlin.

Wendland, H. (1998). Error estimates for interpolation by compactly supported radial basis functions of minimal degree. *J. Approx. Theory*, 93(2):258-272.

Wendland, H. (1999). Meshless galerkin methods using radial basis functions. *Math. Comp. Amer. Math. Soc.*, 68(228):1521-1531.

Zhao, J. and Corless, R.M. (2006). Compact finite difference method for integro-differential equations. *Appl. Math. Comput.*, 177(1):271-288.