Construction of Maximum period Linear Feedback Shift Registers (LFSR)
(Primitive polynomials and linear recurring relations)

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Abstract

Feedback Shift Register (FSR) is generally the basic element of pseudo random generators used to generate cryptographic channel or set of sequences for encryption keys. This type of generator is widely used in stream cipher and communication systems such as C.D.M.A (Code Division Multiple Access) , mobile communication systems, ranging and navigating systems, spread spectrum communication systems.

The objective of the present paper is to propose a method for determining linear recurring sequences generating linear feedback shift register (LFSR) from primitive polynomials (and vice-versa). The linear recurring sequences facilitate the construction of maximum length LFSR. It also insists, in the last part, on the cryptographic security of LFSR and indicates some open problems in the area of nonlinear feedback shift registers (NLFSR) based pseudo random generators.

1 Introduction

Contrary to blocks ciphers algorithms, stream cipher algorithms operate on every unit of the plaintext (encryption of a bit/character at time, bit by bit or character by character) ; the bits are encrypted individually. They are generally faster than blocks cipher algorithms, and have less complex circuits. [19][29][34][59][64][71][72][76][77][78][83][85][88][89][90][92][97][99].

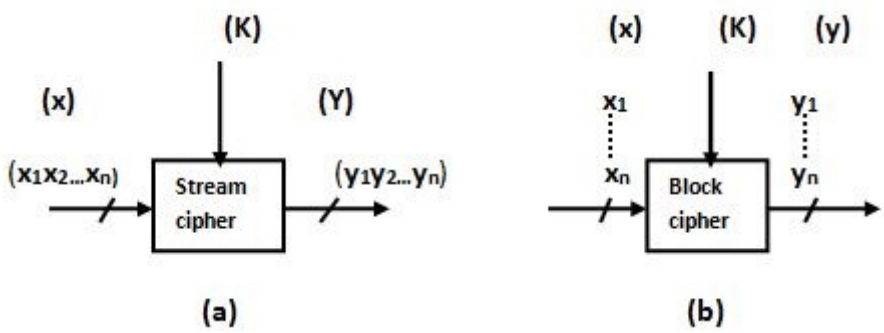

Figure 1: (a) Stream cipher and (b) Block cipher

The stream cipher based on a key generator which generates a bit stream (Key Stream) i.e. a key sequence $K = (k_1, k_2, ..., k_n)$ that combined (XOR function) to bits of plaintext $X = (x_1, x_2, ..., x_n)$ provides ciphertext $Y = (y_1, y_2, ..., y_n)$.

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- encryption equation $y_i = E_{ki}(x_i) = x_i \oplus k_i$
- decryption equation $x_i = D_{ki}(y_i) = y_i \oplus k_i$

(E_{ki}=encryption function and D_{ki}=decryption function)

Encryption is reciprocal: we encrypt as we decrypt.The key stream generator may be regarded as a finite state machine.

- An error in y_i affects only one bit x_i
- The loss or addition of a bit y_i affects all following bits (x_i) after decryption
- if $k_i = 0, \forall i, X = Y$
- if the following key (k_i) is infinite and completely random, one obtain crypto system key-a-time crypto system (One-Time-Pad) also called VERNAM cipher, name of its inventor Gilbert VERNAM (1917) [29][64][71][77][78][80][96][97][101]which is unconditionally secure against a cipher text only attack, the cryptogram contributes no information about the plaintext.

Generally, the stream cipher is based on the same principle as the "One-Time-Pad" with the only difference that it requires a real random sequence which cannot be produced unless you know the whole sequence.

In default, pseudo-random key sequences generated by a pseudo-random generator is therefore used[38][39][50][85][102]. Good pseudo-random sequence is one for which, knowing a portion of the sequence, it is extremely difficult in practice to determine the rest of the sequence[15]. A classic method for generating a pseudo random sequence[45][46]is to use a feedback shift register[cf.paragraph 2].

The bit stream (Stream Key) or sequences of keys generated by the key stream generator constitutes a cryptographic chain.

The stream cipher is classified into two (2) categories:

- Synchronous stream cipher:

  In synchronous stream cipher, the flow of bits of the key stream is generated independently both of the bits of the plaintext and the bit stream of the cipher text.The sender and receiver must be synchronized i.e. use the key stream and be in the same condition that the decryption can be done. If there's a loss or addition of bits, the decryption fails. However, changing a bit in the transmission does not interfere in the decryption of the following bits.

  Examples[29][89]: Output Feedback Mode (OFB) for block cipher systems and CTR mode(Counter Mode) are examples of synchronous stream cipher.

- Self synchronizing stream cipher or asynchronous stream cipher:

  In self-synchronizing stream cipher (or asynchronous stream cipher) each bit of the stream generated by the key generator is a function of a fixed number of bits of the preceding cryptogram. In this method, the generators are synchronized automatically.

If some encrypted bits are lost or added in the cipher text,self synchronization is always possible. However, the system is subject to error propagation, and similarly, a modification of the cipher text by the descriptor can lead in an incorrect decryption of several bits.

Example[89]:Cipher Feedback Mode (CFB) transforms block cipher into self asynchronous stream cipher.

The two (2) methods of stream cipher mentioned above are described in detail, in [19][29][40][71][78][80][82] [85][89].

2 LFSR(Linear Feedback Shift Register)

An example of this type of generator is the FSR (Feedback Shift Register=Shift Registers+a feedback function) .

2.1 Definition:[9][23][45][75][89]

- A flip-flop (position on a delay line or other memory device) is an electronic device capable of storing binary information (bits 0 and 1).
- A shift register of length (n) consists of n-flops interconnected such that the binary state of the memory cell of rank (i) is transmitted to the memory cell of rank $(i+1)$ when a clock signal is applied to the all flip-flops. Each flip-flop may be seen as a stage of the register. The binary information of the last stage is always accessible physically.

A shift register is then constituted of:

- An input which, in shift mode, will advance the bit of a flip-flop to a next flip-flop.
- (n) flops constituting the register stages.
- And an output.

Example of a shift-register of 11 stages:

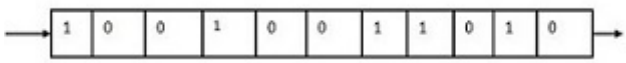


Figure 2: 11 bits LFSR

Most pseudo-random generators are constructed using feedback shift registers (Example:eSTREAM project running from 2004 to 2008 to choose new standard stream ciphers: Sosemanuk, Grain, Mickey, Trivium) . [1][7][8][9][45][67][68][84]

The Feedback Shift Registers constitute the base of pseudo-random generators used for generation of encryption key. This type of generator is largely used in stream cipher.

A Feedback Shift Register (FSR) of size (n) is an automate constructed by a boolean function (f) (ref: Definition 2.2) and a function (F) both with n variables over a field $GF(p)$ such that $F : \{0,1\}^n \to \{0,1\}^n$ (often $p = 2$ for binary field where $p = 2^w$ for some extension field of the binary field) .

$$F(x_1, x_2, ..., x_n) = (x_2, x_3, ..., x_n, f(x_1, x_2, ..., x_n)) \tag{1}$$

- F which is the function of the next state, gives the new state of the FSR from the prior state;
- and (f) which is the feedback function calculates the $n-th$ term of the next state;
- If $(x_1, x_2, ..., x_n)$ is the initial state then the application of (f) and (F) give the state sequence:

$$\begin{aligned}
F(x_1, x_2, ..., x_n) &= (x_2, x_3, ..., x_n, x_{n+1}); x_{n+1} = f(x_1, x_2, ..., x_n) \\
F(x_2, x_3, ..., x_{n+1}) &= (x_3, x_4, ..., x_{n+1}, x_{n+2}); x_{n+2} = f(x_2, x_3, ..., x_{n+1}) \\
F(x_3, x_4, ..., x_{n+2}) &= (x_4, x_5, ..., x_n, x_{n+1}, x_{n+2}, x_{n+3}); x_{n+3} = f(x_3, x_4, ..., x_{n+2}) \\
&\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

The output sequence generated by the FSR:

$$(x_i)_{i \in N} = (x_1, x_2, ..., x_n, x_{n+1}, ...) \tag{2}$$

satisfies the relation of recurrence:

$$x_{i+n} = f(x_i, x_{i+1}, x_{i+2}, ..., x_{i+n-1}) \tag{3}$$

If the feedback function (f) is linear (ref:2.2 Definition) , the FSR is called Linear Feedback Shift Registers(LFSR).Otherwise, it is called Nonlinear Feedback Shift Register (NLFSR) .[36]

2.2 Definition:[1][9][10][19][23][28][29][45][46][57][59][65][66][75][77][82][87][89][90][91][96][97][98]

- A linear feedback shift register of n bit-length (LFSR n-bits) is composed of two parts:
  - One shift register containing a sequence of n bits $(x_1, ..., x_n)$ arranged from left to right which is the initial state of the register;
  - And, a linear feedback function $f(x_1, x_2, ..., x_n)$

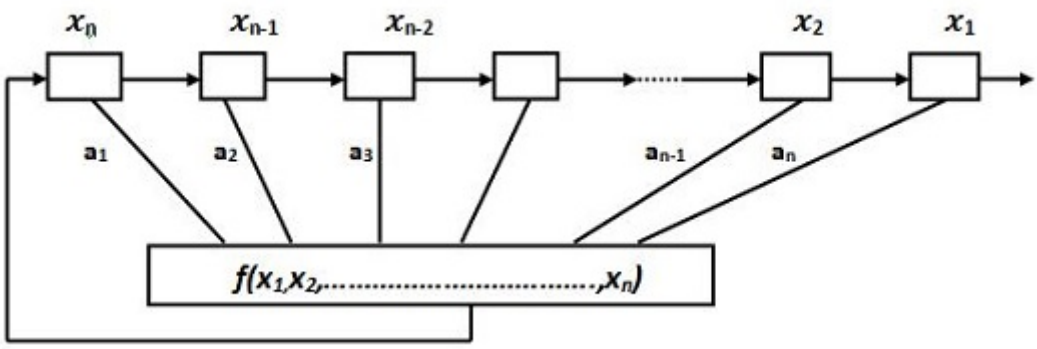


Figure 3: General Scheme of Linear Feedback Shift Register

  - The registry is called by its acronym: LFSR(Linear Feedback Shift Register).
  - At periodic intervals determined by the clock, the content of the stage (i) is transferred into the stage $(i+1)$: a bit is required at any time, and all the bits in the register are shifted forward.
  - The new left most bit is obtained from the other bits in the register with the feedback function $f(x_1, x_2, ..., x_n)$;
  - Output register is 1 bit; the sequence generated is called derivation sequence (output stream) .
  - The period of the LFSR is the length of the sequence generated before it repeats (ref. Definition 2.1).
  - The feedback function $f(x_1, x_2, ..., x_n)$ is such that:

$$\begin{aligned} f(x_1, x_2, ..., x_n) &= a_n x_1 + a_{n-1} x_2 + ... + a_1 x_n \\ &= \sum_{i=1}^{n} a_i x_{n-i+1} \end{aligned} \tag{4}$$

where $a_i = 0$ or $1, \forall i \ 1 \leq i \leq n$, and addition (XOR operation) is over GF(2).

  - $f : GF(2^n) \rightarrow GF(2)$
  - f is a boolean function of (n) variables[16][17][31][29][47][79][82];

– There are 2^{2^n} different boolean functions for (n) variables given.
– The sequence produced by the LFSR satisfy the relation of linear recurrence:

$$x_{n+j} = \sum_{i=1}^{n} a_i x_{n+j-i} \Leftrightarrow x_{n+j} = \sum_{i=0}^{n-1} a_{i+1} x_{n+j-i-1} \tag{5}$$

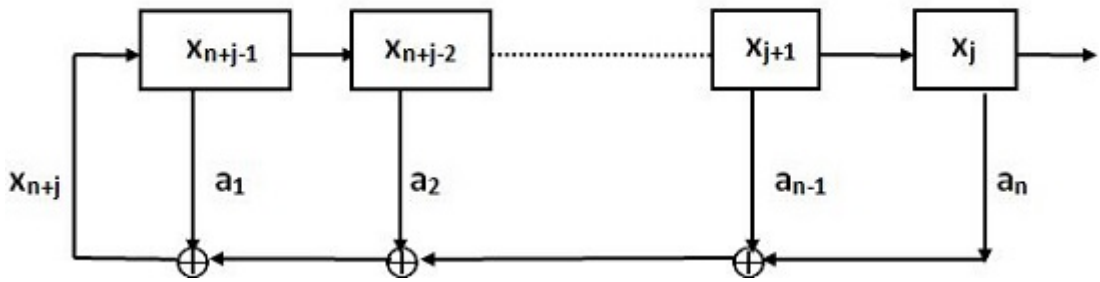


Figure 4: A scheme of LFSR

and the matrix A associated with the linear mapping is:

$$[GF(2)]^n \to [GF(2)]^n$$

$$\begin{bmatrix} x_j \\ x_{j+1} \\ \vdots \\ x_{n+j-2} \\ x_{n+j-1} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{j+1} \\ x_{j+2} \\ \vdots \\ x_{n+j-1} \\ x_{n+j} \end{bmatrix} \tag{6}$$

$$A = \begin{pmatrix} 0 & 1 & 0\ldots & 0 & 0 \\ 0 & 0 & 1\ldots & 0 & 0 \\ & & & & \\ & & & & \\ 0 & 0 & 0\ldots & 0 & 1 \\ a_n & a_{n-1} & \ldots & a_2 & a_1 \end{pmatrix} \tag{7}$$

- Such the configuration, which we'll be interested, is called FIBONACCI's configuration (Fibonacci generator)" [3][29][46][66][82][89]. It is efficient in hardware as it requires only one n-bits LFSR and XOR operations although inefficient in software implementation (LFSR in mode Fibonacci or External-XOR LFSR)unlike its other counterpart called GALOIS configuration (LFSR in mode Galois or internal-XOR LFSR) "discussed in[29][46][82][89].

- The Fibonacci generator or External-XOR LFSR is based on the Fibonacci's sequences modulo the maximum value desired:

$$x_n = (x_{n-1} + x_{n-2}) \pmod m \text{ with } x_0 \text{ and } x_1 \text{ in input} \tag{8}$$

Alternatively, we can use this form called "generalized Fibonacci" recurrences to generate pseudo-random numbers[3][18][61]:

$$x_n = \pm(x_{n-s} \pm x_{n-k}) \pmod m \text{ with } x_0....x_{k-1} \text{ in input as words of w bits; generally } m = 2^w. \tag{9}$$

The quality of the generator depends of the coefficients (k),(s) which must be carefully chosen and the values used for the initial state of the generator. This generator is against very simple to implement and consumes little resources.

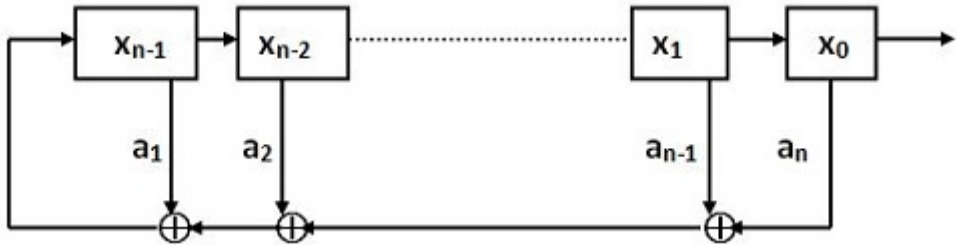


Figure 5: LFSR in mode Fibonacci

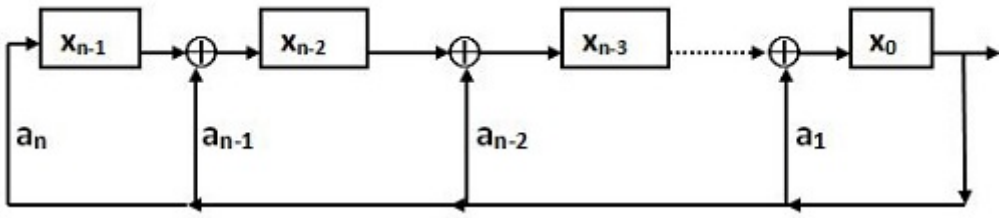


Figure 6: LFSR in mode Galois

- We must differentiate this one from the linear congruential generator[35][60][66][77][82][89][97]that produces pseudo-random sequences of the form:

$$x_n = (ax_{n-1} + b) \pmod{m}, \text{ where } a,\ b \text{ and } m \text{ are integers.} \tag{10}$$

  x_n is the (n)-th bit in the sequence; x_0= the seed, the period of the generator is less than m.

  If a, b, and m are carefully chosen, then the generator will be said to be "maximum period (m) (e.g.if $b \wedge m = 1, (b)$ and (m) are coprime) , if $b = 0$, the generator is said to be homogeneous congruential multiplicative.[66][35]

  The linear congruential generators are fast and require little bit operations, but it has been proved that they cannot be used in cryptography for stream cipher. Indeed, they can be predicted and therefore are decryptable[89]. It is valid for:

  - The Quadratic generators:

$$x_n = (ax_{n-1}^2 + bx_{n-1} + c) \pmod{m} \tag{11}$$

  - The cubic generators:

$$x_n = (ax_{n-1}^3 + bx_{n-1}^2 + cx_{n-1} + d) \pmod{m} \tag{12}$$

  discussed in[89] who notes that the combination of linear congruential generator providing long periods were not also proved safe cryptographically.

- The LFSR are used extensively in stream cipher because they are easily implemented in hardware as well as in software. Referring to the above definitions, it is possible to generalize the LFSR, in any finite $GF(p)$. On the software aspect, it is used finite field of the form $GF(2^n)$ with $n = 8, 32, 64, ...$

2.3 Examples

Example 1: One maximal-period n-bits LFSR

A maximal-period n-bits LFSR on $GF(2)$with maximal period $T = 2^n - 1$ is a register which can theoretically generate a pseudo-random sequence of length $T = 2^n - 1$ bits before the repetition(and not 2^n the null sequence(000...000) is not considered. The resulting output sequence is called an "m-sequence".

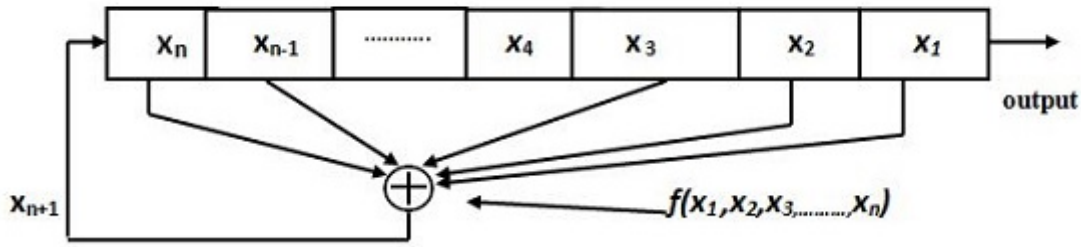
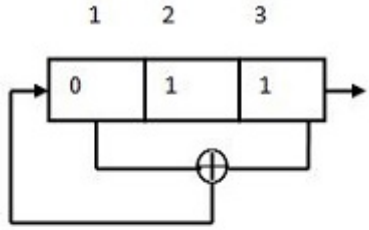


Figure 7: n-bits LFSR

Example 2: Maximal period 3-bits LFSR: There are two possible loops:

- the stages 1 and 3: $x_{n+1} = x_n + x_{n-2} \pmod 2$ (recurrence equation)
- the stages 2 and 3: $x_{n+1} = x_{n-1} + x_{n-2} \pmod 2$ (recurrence equation)
- The loop 1 and 2 is forbidden: the LFSR loop after two steps as shown in figure 10 (no maximal period)

Loop 1 and 3: Maximal period. The maximal period is: $T = 2^3 - 1 = 8 - 1 = 7$.
The recurrence equation is: $x_{n+1} = x_n + x_{n-2} \pmod 2$.



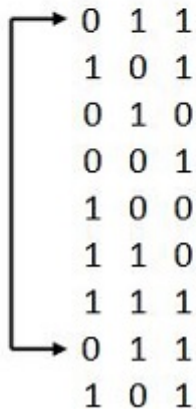
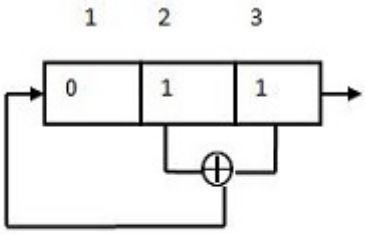


Figure 8: 3-bits LFSR

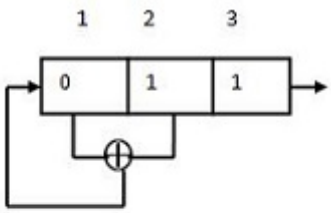
Key stream: 1101001
Loop 2 and 3: Maximal period
The recurrence equation is: $x_{n+1} = x_{n-1} + x_{n-2} \pmod 2$



0	1	1
0	0	1
1	0	0
0	1	0
1	0	1
1	1	0
1	1	1
0	1	1
0	0	1

Figure 9: 3 bits LFSR

Key stream: 1100101
Loop 1 and 2:
The recurrence equation is: $x_{n+1} = x_n + x_{n-1} \pmod 2$



1	0	1
1	1	0
0	1	1
1	0	1

Figure 10: 3-bits LFSR

Example 3: Maximal period 4-bits LFSR
The maximal period is $T = 2^4 - 1 = 15$ with:
Loops 1 and 4: Maximal-period
The recurrence equation is: $x_{n+1} = x_n + x_{n-3} \pmod 2$

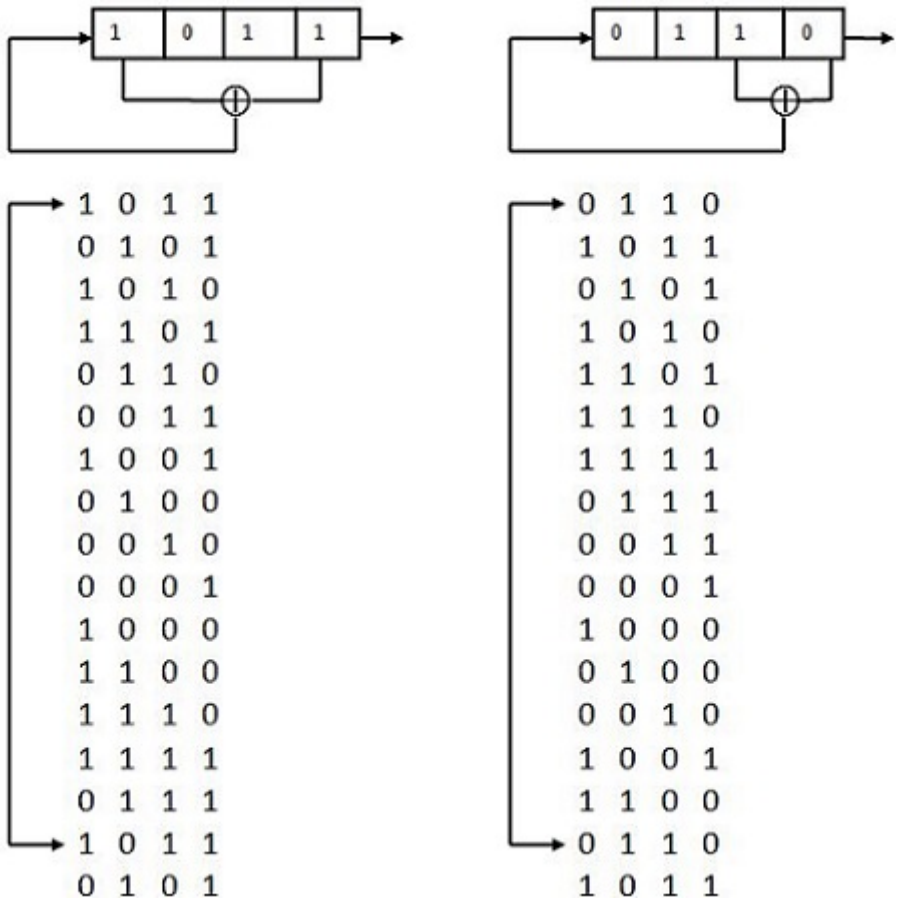


Figure 11: 4-bits LFSR

Keys stream: 110101100100011 and 011010111100010
Loop 3 and 4: Maximal period
The recurrence equation is: $x_{n+1} = x_{n-2} + x_{n-3} \pmod 2$
Example 4: maximal period 6-bits LFSR
The maximal period is obtained by: $T = 2^6 - 1 = 63$

- the loop 1 and 6: $x_{n+1} = x_n + x_{n-5}$;
- the loop 5 and 6: $x_{n+1} = x_{n-4} + x_{n-5}$;

(Loop 5 and 6: Maximal period)

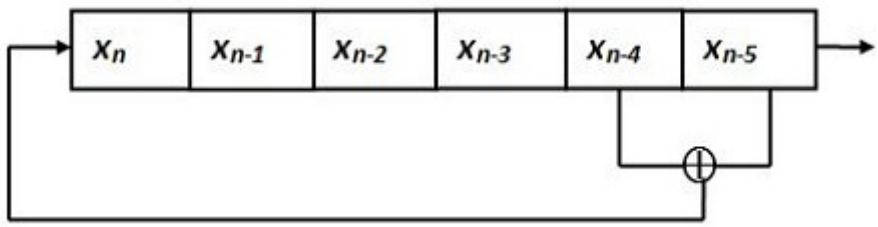


Figure 12: 6-bits LFSR

Example 5: maximal period 31-bits LFSR
The maximal period is obtained by: $T = 2^{31} - 1 = 2.147.483.647$

- the loop 3 and 31: $x_{n+1} = x_{n-2} + x_{n-30} \pmod 2$;
- the loop 28 and 31: $x_{n+1} = x_{n-27} + x_{n-30} \pmod 2$;

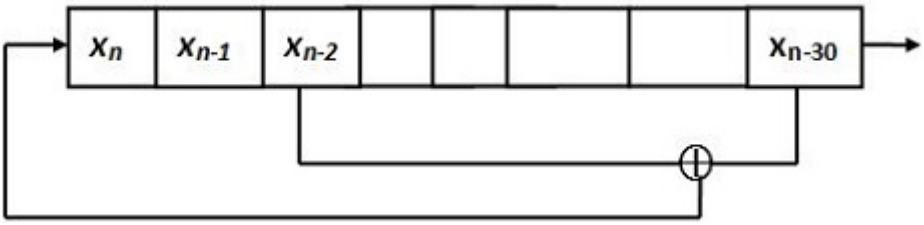


Figure 13: 31-bits LFSR

2.4 Theorem: [1][9][23][46][59][75][89][91]

For an n-bits LFSR be maximal period,it is necessary that the polynomial formed from derivation sequence must be a primitive polynomial of degree (n) in $GF(2)$.

- We thus associate with an n-bits LFSR, a primitive polynomial generator:

$$f(x) = 1 + \sum_{i=1}^{n} a_i x^i = 1 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n \tag{13}$$

- A primitive polynomial of degree (n) is an irreducible polynomial of degree (n) which divides $(x^{2^n-1} + 1)$.

Example: the polynomial $f(x) = x^3 + x + 1$ of degree 3 is primitive over $GF(2)$, it divides $x^7 + 1$, $(T = 2^n - 1 = 2^3 - 1) \longrightarrow x^7 + 1 = (x^3 + x + 1)(x^4 + x^2 + x + 1)$.

- if any polynomial $f(x)$ associated to a LFSR is primitive over $GF(2)$ then any non-zero initial state produces a sequence of maximal period $T = 2^n - 1$.

2.5 Definition 3:

Let f be a polynomial $f(x)$ in $F_2[X]$. Its order, denoted $ord(f)$ is the smallest integer (t) such that $x^t \equiv 1 \bmod f(x)$.

2.6 Definition 4:

Let $f(x)$ be an irreducible polynomial of degree (n) in $F_2[X]$. It is primitive if its order is$(2^n - 1)$.

So, we want to build an optimal n bit LFSR (in relation to the period after production), we must ensure that the feedback polynomial chosen is of degree n and primitive.

We will be sure to obtain a maximal period, but taking the precaution of using a non-zero initial state.

Another advantage of feedback primitive polynomials is the statistical quality of sequence produced.

2.7 Definition 5:[9]

Let $f(x)$ be an irreducible polynomial over $([GF(2^n)])^*$. It is called primitive if one of its roots generates the multiplicative subgroup $[GF(2^n)]$, with $[GF(2^n)] = F_2[X]/f(x)$(polynomials reduced modulo$f(x)$). First recall that the multiplicative subgroup of a finite field is cyclic. In other words,$\forall \alpha \in [GF(2^n)]$, we have $\alpha^{2^n-1} = 1$.

Let (α) a root of (f), then we have $f(\alpha) = 0$. If α generates the multiplicative group, the elements $\alpha, \alpha^2, \alpha^3 ... \alpha^{2^n-1}$correspond to all nonzero elements of the field, and there are $2^n - 1$ distinct elements $(\alpha^{2^n-1} = 1$ since $\alpha \in ([GF(2^n)])^*$ cyclic).

2.8 Definition 6:[9][91]

In fact, we can define the order of an element α as the smallest (t) such that $\alpha^t = 1$. What we want to know is if the order of α is equal to $2^n - 1$ ($f(x) =$ primitive polynomial) or not ($f(x) =$ non-primitive polynomial); it must be remembered that for an n-bit LFSR, $(2^n - 1)$ and $f(x)$ divides $(x^{2^n-1} + 1)$.

Example: Let $f(x) = x^3 + x + 1$ irreducible, and (α) is as $f(\alpha) = 0 \Rightarrow \alpha^3 + \alpha + 1 = 0 \Rightarrow \alpha^3 = \alpha + 1$.
Let us determine the powers of α: $\alpha^1 = \alpha$
$\alpha^2 = \alpha^2$
$\alpha^3 = \alpha + 1$
$\alpha^5 = \alpha^3 + \alpha^2 = \alpha^2 + \alpha + 1$
$\alpha^6 = \alpha^3 + \alpha^2 + \alpha = \alpha^2 + 1$
$\alpha^7 = \alpha^3 + \alpha = 1 \Rightarrow \alpha^{2^3} - 1$ with $T = 7$.
We can conclude that the polynomial $f(x) = x^3 + x + 1$ is primitive.

3 Primitive feedback polynomials and linear recurrences for constructing maximal-period LFSR

With the primitive polynomial, we can identify the linear recurrence equation associated to the $n-$bits LFSR and vice versa.

Let the following register and the primitive polynomial defined in paragraph 2.4 (13):

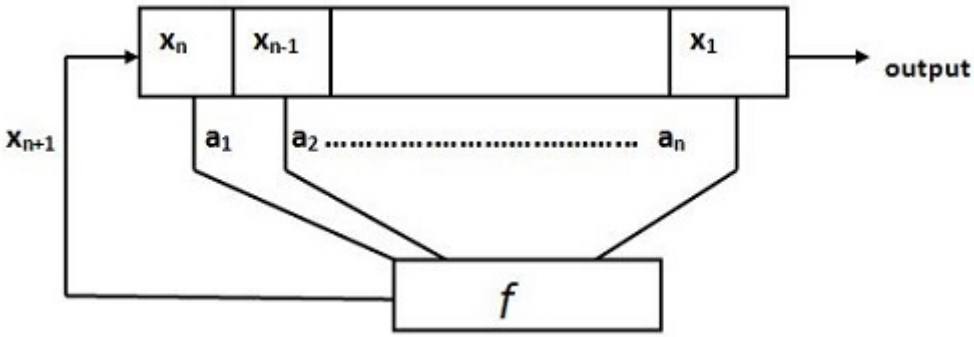


Figure 14: n-bits LFSR

Then, our linear recurrence can be expressed in this form:

$$\begin{array}{rcl} x_{n+1} & = & a_1x_n + a_2x_{n-1} + ... + a_nx_1 = \sum_{k=1}^{n-1} a_kx_{n-k+1} \\ \vdots & & \vdots \qquad\qquad\qquad \vdots \\ x_{n+i} & = & a_1x_{n+i-1} + a_2x_{n+i-2} + ... + a_nx_i = \sum_{k=1}^{n} a_kx_{n-k+i} \end{array}$$

We will consider in our study, the simplest form corresponding to the single equation:

$$x_{n+1} = f(x_1, x_2, ..., x_n) = \sum_{k=1}^{n} a_kx_{n-k+1} \tag{14}$$

3.1 Maximal Period

In the case whether the polynomial generator is primitive, for all initial state of n non-zero bits, T is the maximal period: $T = 2^n - 1$.

3.2 Identification equation

With our primitive polynomial and the linear recurrence, the identification equation is:

$$\begin{aligned} f(x) &= 1 + \sum_{k=1}^{n} a_k x^k = 1 + a_1 x + a_2 x^2 + ... + a_n x^n \\ x_{n+1} &= \sum_{k=1}^{n} a_k x_{n-k+1} \end{aligned} \tag{15}$$

The identification equation help us to determine the coefficients a_k and therefore the recurrence equation from the polynomial(and vice versa).

$$x_{n+1} \Leftrightarrow f(x) \tag{16}$$

1. Application to 4-bits LFSR:

   Let $f(x) = x^4 + x + 1$

   The maximal period: $n = 4$ and $T = 2^n - 1 = 2^4 - 1 = 15$.

   The identification equation:

$$\begin{aligned} f(x) &= x^4 + x + 1 \\ &= 1 + \sum_{k=1}^{4} a_k x^k \\ &= 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \end{aligned} \tag{17}$$

   We can identify then: $a_3 = a_2 = 0$ and $a_4 = a_1 = 1$.
   But $x_{n+1} = \sum_{k=1}^{4} a_k x_{n-k+1} = a_1 x_n + a_2 x_{n-1} + a_3 x_{n-2} + a_4 x_{n-3}$, then recurrence equation is:

$$\begin{aligned} x_{n+1} &= a_1 x_n + a_4 x_{n-3} \pmod 2 \\ x_{n+1} &= x_n + x_{n-3} \pmod 2 \end{aligned} \tag{18}$$

   we obtain the following maximal-period 4 bits LFSR:

Figure 15: 4 bits LFSR

Remark: If we make this change of variable $x = \dfrac{1}{x}$, we have another possible register Indeed if $f(x)$ primitive, $g(x)$ $=x^n f(\frac{1}{x})$ is also primitive. Assuming $f(x) = x^n + x^s + 1 \Rightarrow$ $g(x)$ $=$x$^n(\frac{1}{x}) = x^n(\dfrac{1}{x^n} + \dfrac{1}{x^s} + 1) = x^n + x^{n-s} + 1$ which is also primitive (reciprocal polynomial).

The two polynomials can be used to build register for applications. (More generally $f(x) = 1 + \sum_{k=1}^{n} a_k x^k$ and $g(x)$ $=x^n + \sum_{k=1}^{n} a_k x^{n-k}$)

With the polynomial$f(x) = x^4 + x + 1$, we have $g(x)$ $=x^4 + x^3 + 1 = 1 + \sum_{k=1}^{4} a_k x^k = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$.

By using the identification equation, we have:

- $a_4 = a_3 = 1$ and $a_2 = a_1 = 0$
- $x_{n+1} = \sum_{k=1}^{4} a_k x_{n-k+1} = a_1 x_n + a_2 x_{n-1} + a_3 x_{n-2} + a_4 x_{n-3}$
- $\Rightarrow x_n = x_{n-2} + x_{n-3}$

We obtain then the following maximal-period 4 bits LFSR:

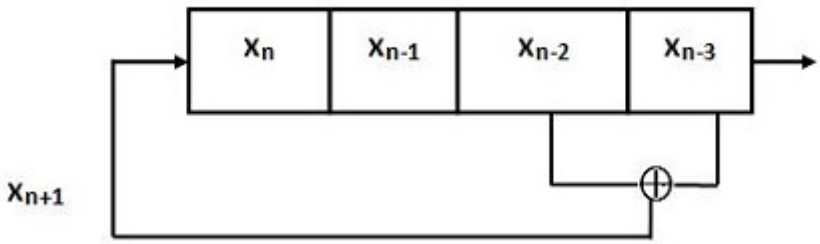


Figure 16: 4-bits LFSR

2. Application to 35-bits LFSR
   Wether: $f(x) = x^{35} + x^2 + 1$

   - maximal period:
     $T = 2^n - 1 = 2^{35} - 1$
   - Identification equation:

$$\begin{aligned} f(x) = x^{35} + x^2 + 1 &= 1 + \sum_{k=1}^{n} a_k x_k \\ &= 1 + a_1 x + a_2 x^2 + ... + a_{34} x^{34} + a_{35} x^{35} \end{aligned} \tag{19}$$

   All $a_k = 0$ except $a_2 = a_{35} = 1$
   - $x_{n+1} = \sum_{k=1}^{35} a_k x_{n-k+1} = a_1 x_n + a_2 x_{n-1} + ... + a_{35} x_{n-34}$
   - By identifying, it comes:

$$x_{n+1} = a_2 x_{n-1} + a_{35} x_{n-34} \tag{20}$$

$$x_{n+1} = x_{n-1} + x_{n-34} \tag{21}$$

We have this maximal-period 35 bits LFSR:

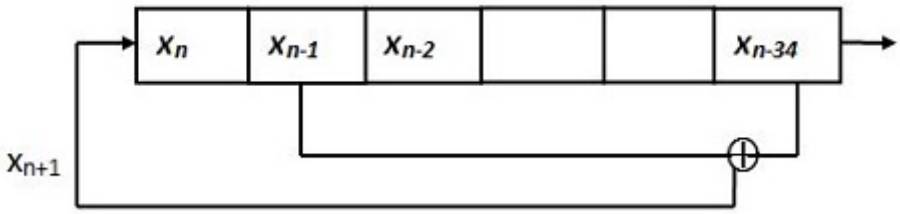


Figure 17: 35 bits LFSR

With the usual change of variable then:
$f(x) = x^{35} + x^2 + 1 \Rightarrow g(x) = x^{35} + x^{33} + 1$ is also primitive.

- Identification equation:
  $g(x) = x^{35} + x^{33} + 1 = 1 + \sum_{k=1}^{35} a_k x^k = 1 + a_1 x_+ a_2 x^2 + ... + a_{33} x^{33} + a_{34} x^{34} + a_{35} x^{35}$
  All the $a_k = 0$ except $a_{35} = 1$ and $a_{33} = 1$

$$
\begin{aligned}
x_{n+1} &= \sum_{k=1}^{35} a_k x_{n-k+1} \\
&= a_1 x_n + a_2 x_{n-1} + ... + a_{33} x_{n-32} + a_{34} x_{n-33} + a_{35} x_{n-34} \\
x_{n+1} &= a_{33} x_{n-32} + a_{35} x_{n-34} \\
x_{n+1} &= x_{n-32} + x_{n-34}
\end{aligned}
$$

3. The maximal-period 35 bits LFSR:

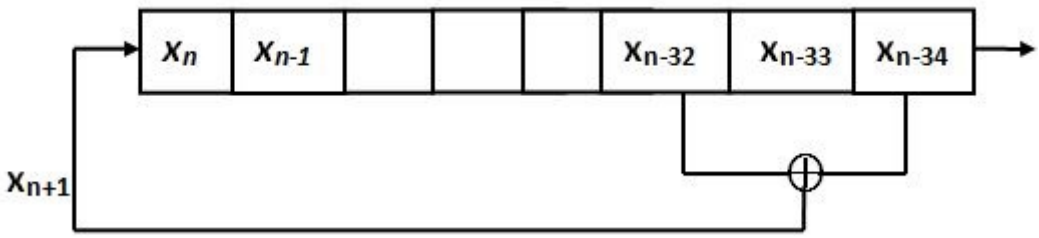


Figure 18: 35 bits LFSR

Finally, it should be noted that[48][66][103][104] recall some works that has been done on binary primitive polynomials and give tables of polynomials available.
However, for the realization of LFSR, it is strongly recommended to use primitive polynomials with non-zero coefficients(dense primitives polynomials) rather than polynomials among which most of the coefficients are zero and which are weak cryptographically.[89]

4 Cryptographic security

4.1 Linear complexity

In terms of cryptographic security, the use of a single LFSR is not sure because this LFSR is predictable:

- And if we know n consecutive bits produced by an n-bits LFSR and the primitive polynomial associated, we can deduce the $(n+1)$-th bit range produced by the register;

- If we also know $(2n)$ consecutive bits produced by an n-bits LFSR without knowing the polynomial primitive associated, we can find this polynomial by the Berlekamp-Massey algorithm[11] [46][65][82][91][93][98].

This algorithm permits us to determine the linear complexity of a random sequence i.e. the length of the smallest LFSR that can generate it(also called linear span)[27][82]. In 1969, James L. Massey [65] proved, in fact, the algorithm proposed in 1967 by Ralph Elwyn Berlekamp for decoding BCH codes[11] also allows the possibility to find the smallest LFSR generating a given sequence[84] and gives a range of results on the linear complexity of random sequences.

4.2 Recommendations-Perspectives

Pseudo-random generator based on FSR used to generate keys must have the following characteristics: [85]

- A long period
- A large linear complexity
- Good statistical properties

As we noted above, the major advantage of LFSR is the ease of hardware and software implementation coupled with their good mathematical conception. However, LFSR, used alone, are not safe an account of their linearity which is exploited to build cryptanalytic attacks foremost among them the Berlekamp-Massey algorithm.

To strengthen cryptographic security of the LFSR generators, we use LFSR generators more complex using nonlinear boolean functions(cryptographic boolean function)[16][17][31][29][47][79][82]which must have certain properties(high algebraic degree, high linear complexity, high non-linearity and high correlation immunity) that can destroy the linearity:

- Non linear combination generators[20][41][43][85][100]: A keystream generator on which the output of several LFSR are combined by a linear function[17][47][85]

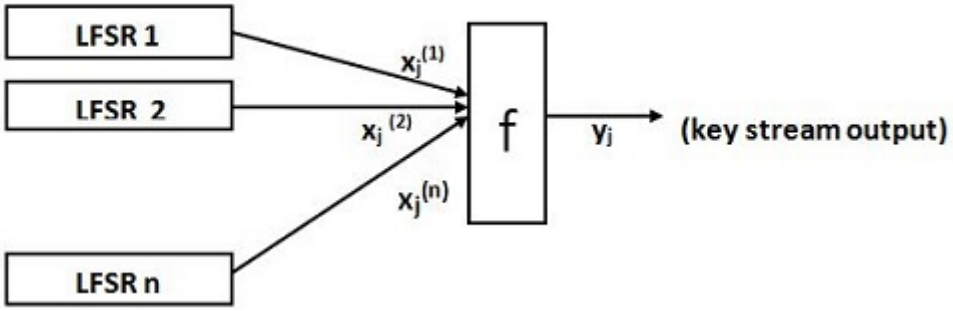


Figure 19: Non linear combined LFSRs

- Non linear filter generator[21][42][85]:A keystream generator consisting of a single LFSR and a nonlinear function (also called Nonlinear filtering function) whose inputs are taken from some shift register stages to produce the output.

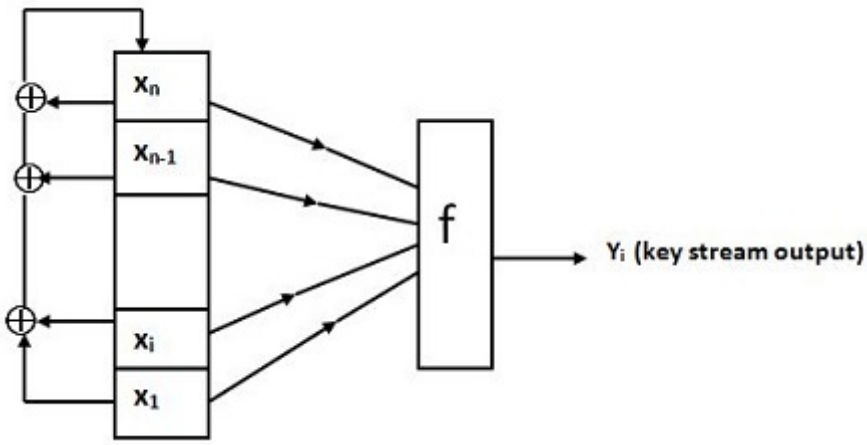


Figure 20: Non linear filter generator

- Clock-controlled generator[12][22][85]: A keystream generator in which an LFSR is used to determine which output symbols of second LFSR are used as the final output.

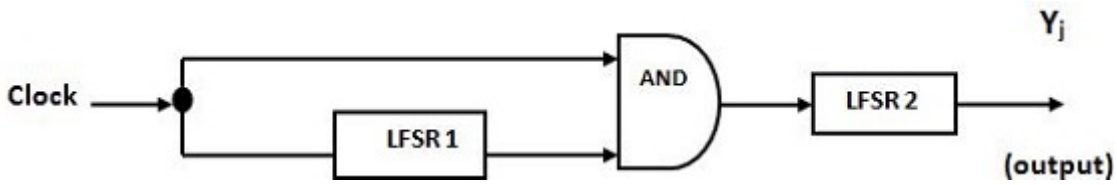


Figure 21: clock control

- It is also more than useful to mention the constructions concerning:
  - The shrinking generated invented in 1993, by D. Coopersmith, H. Krawczys and Y. Mansour[14][24].
  - the self-shrinking generator invented in 1994 by W. Meier and O. Staffelback[70].

But the additional security measures have not so far allowed to shelter attacks:

- Exhaustive attacks;
- Time memory Data Tradeoff attacks[6][13][51];
- Correlation attacks[39][44][51][52][68][69][85][86][94]
- Algebraic attacks[25][26][30][51][52][62][63][68]
- Side channel attacks[53][58][68][95]
- Distinguishing attacks[49][68]

At present, the researches are directed to new classes of feedback Shift registers based notably on the algebraic rings, implying new methods of cryptanalysis and new security measures[27][46][55][56].
In fact, important studies have been conducted in the field of Nonlinear feedback Shift registers(NLFSR), both from the point of view of design as attacks, and which resulted:

- since 1993(by Andrews Klapper and Mark Goresky)to feedback with carry shift registers(FCSR)[46][54][55][56]in Fibonacci mode or Galois mode, with as mathematical basis, the ring of $N-$Adic integers instead of the ring of formal series used for the LFSR.

However, in spite of their large linear complexity, they are susceptible to attacks by the rational approximation algorithm[56]which is similar to that of Berlekamp-Massey.

Therefore they could be coupled with LFSR in the design of pseudo random generators.

- to vectorial feedback with cary shift registers(VFCSR), vector design of the FCSR whose analysis has been extended to finite fields $GF(p^n)$.[2]
- to Filtered feedback with carry shift registers(F-FCSR), design of FCSR to counter the attack by the rational approximation algorithm.[4][5][56]
- to Algebraic feedback shift register(AFSR)when the mathematical basis is $\pi-$Adic ring(not specified as $\pi-$Adic numbers are generalizations of formal series and $N-$Adic integers). LFSR and FCSR are special cases of the AFSR.[46]

It is useful to take a look also on the theory of stability of stream cipher cryptosystem i.e the resistance of such systems to small variations in some of their parameters as regard in particular the linear complexity and nonlinear boolean functions used [27][33][34]:

- For additive synchronous stream cipher, there are already techniques of control of the stability of the linear complexity. But, the problem of the stability of the local linear complexity seems for the moment difficult to solve (this is an open problem).
- For nonlinear combined registers and nonlinear filtered register, partial results were obtained on certain aspects of the theory of stability, but research should be carried further: a promising field of research.

Other ways of research could be explored in the field of studies made, in particularly, on metric spaces and series [73][74] [32].
Finally, research is also conducted on the registers with Nonlinear Update(RNLUs) which are generalization of NLFSRs whose study is theoretical and should be further refined.[37][81]

5 Conclusion:

As indicated at the beginning of the article, the proposed method determines mathematically, from the primitive polynomial, linear recurring relation generating the LFSR (and vice versa), and thus facilitate its construction ; it also helps to establish the corresponding reciprocal primitive polynomial which gives the possibility to build another LFSR as good as the first.

We have a design and a careful choice of maximum length LFSR to use, on the basis of the primitive polynomial, the reciprocal polynomial, and associated linear recurring relations, that do not show the methods used so far where the recurrences are established, without further details, from the primitive polynomial to draw the LFSR.

On the other hand, it seemed important to review the LFSR, and to emphasize their cryptographic security with recommendations in the above paragraph and in highlighting research opportunities in this area.

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