# Approximate solutions for a couple of reaction-diffusion equations with self-diffusion

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***Type of Article***

## Abstract

In this article, a reaction diffusion system with self-diffusion has been studied using homotopy perturbation method, variational iteration method and Finite element method FEM (COMSOL package). The traveling wave solutions for the above system are found and compared numerically. The effect of self diffusion is shown in the traveling wave solution. Also, it was shown that the solution of finite element method and homotopy perturbation method are convergent to each other compared to the variation iteration method.

*Keywords: Reaction-Diffusion system ; Homotopy Perturbation Method; Variational Iteration Method ; Finite Element Method.*

## 1 Introduction

Most of problems and phenomena in different fields of science occur nonlinearly. A reaction diffusion equation has applications in fields of ecology, physics, chemistry and biology [1]. Nonlinearity of reaction or diffusion term describes phenomena in the above fields. A reaction diffusion system (1.1,1.2) has nonlinearity in both terms,

$$u_t = S_1(u)u_{xx} + F(u,v) \tag{1.1}$$

$$u_t = S_2(v)v_{xx} - G(u,v), \tag{1.2}$$

subject to the initial conditions
$u(x,0) = f(x)$ and $v(x,0) = g(x)$,

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and the boundary conditions
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0,$
which are are zero flux boundary conditions. The functions $u$ and $v$ for example represents in ecology two interactions species, $F(u,v) = u(1-u-v)$ and $G(u,v) = -uv$ are the reaction terms. The rest is the diffusion term, the term $1-u$ in first equation is a logistic growth, and $-uv$ describes a competition between two species in both equations. The self diffusion in (1.1,1.2) is the quadratic functions $S_1(u)$ and $S_2(v)$ multiplying by the diffusion term. The term self-diffusion implies the movement of individuals from a higher to lower concentration region (see for example [3, 2]). When $S_1(u) = S_2(v) = 1$, equation (1.1,1.2) reduced to a simple Noyes-Field model for the Belousov-Zhabotinskii reaction which is studied widely in [4]. Equilibrium solutions are solutions that satisfied $F(u,v) = G(u,v) = 0$, which are here $(u,v) = (0,0)$, $(u,v) = (1,0)$ and $(u,v) = (0,1)$. Traveling wave solution is a solution that connects stable equilibrium solution to other equilibrium solution. We will focus here on the traveling wave that connects stable equilibrium point $(0,1)$ to $(0,1)$.
Finite element package COMSOL has been used widely to find the approximate solutions for non linear partial differential equations which has no exact solutions. This tool give a high accuracy of solutions in one and two dimensions (for more details see [5]). This package is used here to solve (1.1,1.2) in one dimension, so we can compare the results from this method to the two methods below.

The homotopy perturbation method [6], proposed first by He in 1998 and was further developed and improved by He [7, 8]. The method yields a very rapid convergence of the solution series in the most cases. Usually, one iteration leads to high accuracy of the solution. Although goal of Hes homotopy Perturbation method was to find a technique to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems.
The variational iteration method was first proposed by He [11, 9, 10] and was successfully applied to autonomous ordinary differential equations in [12], to nonlinear polycrystalline solids in [13], and other fields. The combination of a perturbation method, variational iteration method, method of variation of constants and averaging method to establish an approximate solution of one degree of freedom to weakly nonlinear system in [14]. The variational iteration method has many merits and has much advantages over the Adomian method [15].
The aim of this paper is to obtain the approximate solutions of coupled systems of reaction-diffusion equations using HPM and VIM . The capability, effectiveness and convenience of this methods are revealed by obtaining the approximate solutions of the models and comparing FEM using COMSOL.

# 2 Analysis of the Homotopy Perturbation Method

In this section, we illustrate the basic idea of this method, first we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{2.1}$$

with the equation of boundary

$$B(u, \frac{\partial u}{\partial n}) \qquad r \in \Gamma \tag{2.2}$$

where $A$ is a general differential operator, $B$ a boundary operator, $f(r)$ is known as an analytical function, and $\Gamma$ is the boundary of the domain $\Omega$. The term $A(u)$ can be divided into two parts which are $L$ and $N$, where $L$ is a linear operator and  is nonlinear term. We can rewrite (2.1) as follows:

$$L(u) + N(u) - f(r) = 0, \qquad r \in \Omega \tag{2.3}$$

Homotopy perturbation structure is shown as follows:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{2.4}$$

where

$$v(r,p):\ \Omega \times [0,1] \to R \tag{2.5}$$

In equation(2.4), $p \in [0,1]$ is an embedding parameter and $u_0$ is the first approximation that satisfies the boundary condition. Thus, the solution of (2.4) can be written as a power series in $p$,

$$v = v_0 + pv_1 + p^2 v_2 + \dots, \tag{2.6}$$

or as an approximate solution,

$$u = \lim_{p \to 0} v = v_0 + v_1 + v_2 + \dots \tag{2.7}$$

It was proved that the series (2.7) is convergent for most of the cases and the rate of convergence depends on $L(u)$ (see for example [6]). We use this method in the section of result to find the solution of the reaction-diffusion system in (1.1) and (1.2).

# 3 Basic idea of variational iteration method (VIM)

To clarify the basic ideas of VIM, we consider the following differential equation

$$Lu + Nu = g(x,t), \tag{3.1}$$

where $L$ is a linear operator defined by $L = \frac{\partial^m}{\partial t^m}, m \in \mathbb{N},\ N$ is a nonlinear operator and $g(x,t)$ is known to be analytic function. The idea of (VIM) is write down a correction functional as follows:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) \\ - g(x,\tau))\, d\tau, \end{aligned} \tag{3.2}$$

where $\lambda$ is a general Lagrangian multiplier [16] and can be defined as,

$$\lambda(x,\tau) = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}, \quad m \geq 1. \tag{3.3}$$

The subscript $n$ indicates the $n$th approximation of and $\tilde{u}_n$ is considered as a restricted variation [17]. We used this method to find the approximate solution for (1.1) and(1.2), and compare it to the other methods.

# 4 Result

In this section we apply FEM COMSOL, HPM and VIM to the problem in order to find the traveling wave solutions for (1.1)-(1.2) when $S_1(u)$ and $S_2(v)$ are non linear terms.
We consider a coupled system of reaction-diffusion equation with self diffusion

$$u_t = u^2 u_{xx} + u(1 - u - v) \tag{4.1}$$

$$u_t = v^2 v_{xx} - uv \tag{4.2}$$

Subject to the initial conditions of:

$$u(x,0) = \frac{e^{kx}}{(1 + e^{0.5kx})^2} \tag{4.3}$$

$$v(x,0) = \frac{1}{(1 + e^{0.5kx})} \tag{4.4}$$

Where $k$ is constant.

## 4.1 HPM solutions of (4.1) and (4.2)

In order to solve equation(4.1) and equation(4.2), using HPM, we construct the following homotopy process for these equations:

$$H(u,p) = (1-p)u_t + p(u_t - u(1-u-v) - u^2 u_{xx}) = 0 \tag{4.5}$$

$$H(u,p) = (1-p)v_t + p(v_t - v^2 v_{xx} + uv) = 0 \tag{4.6}$$

Substituting $v$ from equation(2.6) into equations(4.5) and (4.6) and rearranging based on powers of $p$ terms, we can obtain:

$$p^0 : \begin{cases} \frac{\partial u_0}{\partial t} = 0 \\ \frac{\partial v_0}{\partial t} = 0 \end{cases} \tag{4.7}$$

$$p^1 : \begin{cases} \frac{\partial u_1}{\partial t} - u_0 + u_0 v_0 + u_0^2 - u_0^2 \frac{\partial^2 u_0}{\partial x^2} = 0 \\ \frac{\partial v_1}{\partial t} + u_0 v_0 - v_0^2 \frac{\partial^2 v_0}{\partial x^2} = 0 \end{cases} \tag{4.8}$$

$$p^2 : \begin{cases} \frac{\partial u_2}{\partial t} - u_1 + 2u_0 u_1 + v_0 u_1 + u_0 v_1 \\ \qquad -(u_0^2 \frac{\partial^2 u_1}{\partial x^2} + 2u_0 u_1 \frac{\partial^2 u_0}{\partial x^2}) = 0 \\ \frac{\partial v_2}{\partial t} + u_0 v_1 + v_0 u_1 - (v_0^2 \frac{\partial^2 v_1}{\partial x^2} \\ \qquad +2v_0 v_1 \frac{\partial^2 v_0}{\partial x^2}) = 0 \end{cases} \tag{4.9}$$

Solving equations (4.7)-(4.9), we obtain:

$$u_0 = \frac{e^{kx}}{(1+e^{0.5kx})^2} \tag{4.10}$$

$$v_0 = \frac{1}{(1+e^{0.5kx})} \tag{4.11}$$

$$u_1 = \frac{te^{\frac{3kx}{2}} A}{2\left(e^{\frac{kx}{2}}+1\right)^8} \tag{4.12}$$

$$v_1 = \frac{\frac{1}{4}\left(k^2-4\right)te^{kx} - \frac{1}{4}te^{\frac{kx}{2}} B}{\left(e^{\frac{kx}{2}}+1\right)^5} \tag{4.13}$$

$$u_2 = \frac{t^2 e^{\frac{3kx}{2}} C}{16\left(e^{\frac{kx}{2}}+1\right)^{14}} \tag{4.14}$$

$$v_2 = \frac{t^2 e^{x/2} D}{32\left(e^{x/2}+1\right)^9} \tag{4.15}$$

$$\vdots$$

where
$A = 2k^2 e^{\frac{3kx}{2}} - k^2 e^{2kx} + 8e^{\frac{kx}{2}} + 8e^{\frac{3kx}{2}} + 12e^{kx} + 2e^{2kx} + 2.$

$B = \left(k^2 + 4e^{\frac{3kx}{2}} + 8e^{kx}\right).$

$C = 448e^{6kx} + 2k^2 + 28k^2 e^{\frac{3kx}{2}} - k^2\left(k^2-6\right)e^{5kx}$
$+ 4\left(3k^2+4\right)e^{\frac{kx}{2}} + 14\left(3k^2+64\right)e^{2kx} + 20\left(5k^2+56\right)e^{\frac{5kx}{2}}$
$+ 14\left(5k^2+64\right)e^{3kx} + 4\left(7k^2+32\right)e^{kx}$
$+ \left(-129k^4 - 84k^2 + 128\right)e^{4kx} + 2\left(17k^4 - 8k^2 + 8\right)e^{\frac{9kx}{2}}$
$+ \left(88k^4 - 60k^2 + 448\right)e^{\frac{7kx}{2}}.$

$D = 3e^{x/2} - 45e^{x} + 11e^{\frac{3x}{2}} - 24e^{2x} + 56e^{3x} + 16e^{\frac{7x}{2}} - 1.$

The solution of (4.1) and (4.2), when $p \to 1$, will be as follows:

$$u(x,t) = u_0 + u_1 + u_2 + \dots. \tag{4.16}$$
$$v(x,t) = v_0 + v_1 + v_2 + \dots. \tag{4.17}$$

## 4.2 VIM solutions of (4.1) and (4.2)

In order to solve equation(4.1) and (4.2), using VIM, we construct a correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda((u_n)_\tau - u_n(1-u_n-v_n) - u_n^2(u_n)_{xx})\, d\tau. \tag{4.18}$$
$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda((v_n)_\tau + u_n v_n - v_n^2(v_n)_{xx})\, d\tau. \tag{4.19}$$

In our equation $m = 1$ , then by formula (3.3) $\lambda = -1$ substituting in equations (4.18) and (4.19) we get:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t ((u_n)_\tau - u_n(1-u_n-v_n) - u_n^2(u_n)_{xx})\, d\tau. \tag{4.20}$$
$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t ((v_n)_\tau + u_n v_n - v_n^2(v_n)_{xx})\, d\tau. \tag{4.21}$$

**For** $n = 0, 1, \dots$.
We start with the initial approximations of $u(x,0)$ and $v(x,0)$ given by equations(4.3) and (4.4). Using the iterations (4.20) and (4.21), we can obtain the other components as follows:

$$u_0(x,t) = u(x,0) = \frac{e^{kx}}{(1+e^{0.5kx})^2}. \tag{4.22}$$
$$v_0(x,t) = v(x,0) = \frac{1}{(1+e^{0.5kx})} \tag{4.23}$$

For $n = 0$

$$u_1(x,t) = u_0(x,t) - \int_0^t ((u_0)_\tau - u_0(1 - u_0 - v_0) - u_0^2(u_0)_{xx})\, d\tau,$$

$$u_1(x,t) = \frac{te^{\frac{3kx}{2}}E}{2\left(e^{\frac{kx}{2}}+1\right)^8} + \frac{e^{kx}}{\left(e^{\frac{kx}{2}}+1\right)^2}. \tag{4.24}$$

$$v_1(x,t) = v_0(x,t) - \int_0^t ((v_0)_\tau + u_0 v_0 - v_0^2(v_0)_{xx})\, d\tau,$$

$$v_1(x,t) = \frac{\frac{1}{4}\left(k^2-4\right)te^{kx} - \frac{1}{4}te^{\frac{kx}{2}}F}{\left(e^{\frac{kx}{2}}+1\right)^5} + \frac{1}{e^{\frac{kx}{2}}+1}. \tag{4.25}$$

where
$E = 2k^2e^{\frac{3kx}{2}} - k^2e^{2kx} + 8e^{\frac{kx}{2}} + 8e^{\frac{3kx}{2}} + 12e^{kx} + 2e^{2kx} + 2.$

$F = \left(k^2 + 4e^{\frac{3kx}{2}} + 8e^{kx}\right).$

By the sane way For $u_2(x,t), u_3(x,t), \ldots$.
The solution of (4.1) and (4.2) via VIM will be as follows:

$$u(x,t) = u_n(x,t). \tag{4.26}$$

$$v(x,t) = v_n(x,t). \tag{4.27}$$

The behavior of $u(x,t)$ and $v(x,t)$ obtained by HPM, VIM, and FEM using COMSOL are shown in Figure(1) to Figure(4). It can be deduced that the traveling wave solutions which obtained from the three methods are more consistent when $t = 0.5$. When time is increased, the traveling wave solutions which obtained from HPM and FEM are more consistent, while the solutions we get from VIM are declined and not entirely agreed with the above methods.

## 4.3 Model with and without self-diffusion

In this section, we compare the traveling wave solution of (1.1,1.2) with and without self diffusion, in other wards, when $S_1(u) = S_2(v) = 1$ the model in (1.1,1.2) turns out to be without self-diffusion. COMSOL is used to compare the solution in both cases as shown in figures (5) and (6). It can be seen how self-diffusion effects on disperse of traveling wave solution of $u$ and $v$.

[Figure 1 about here.]

[Figure 2 about here.]

[Figure 3 about here.]

[Figure 4 about here.]

[Figure 5 about here.]

# 5 Conclusion

In conclusion, three powerful numerical methods are used to find the traveling wave solutions of reaction-diffusion model with self-diffusion in one dimension. It was shown that the traveling wave solutions which are gets from FEM COMSOL and HPM are more consistent compare to the solutions get from VIM with variety in times. The effect of the self-diffusion on the solutions is shown and can be deduced how the solutions will disperse and spread out when the degree of nonlinearity in self-diffusion is changed.

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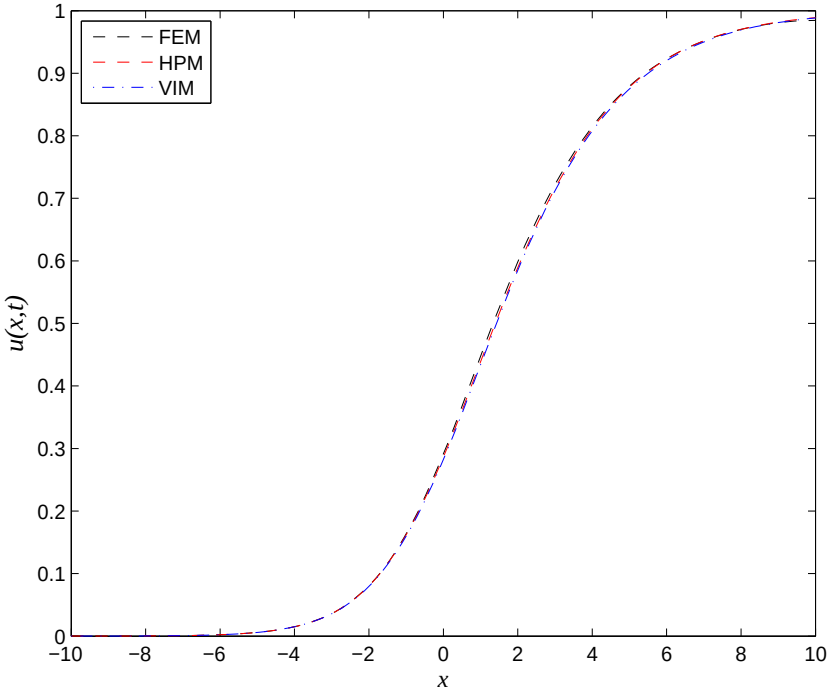


Figure 1: Comparison between the traveling wave solution of $u(x,t)$ using VIM, HPM and COMSOL, when $t = 0.5$.

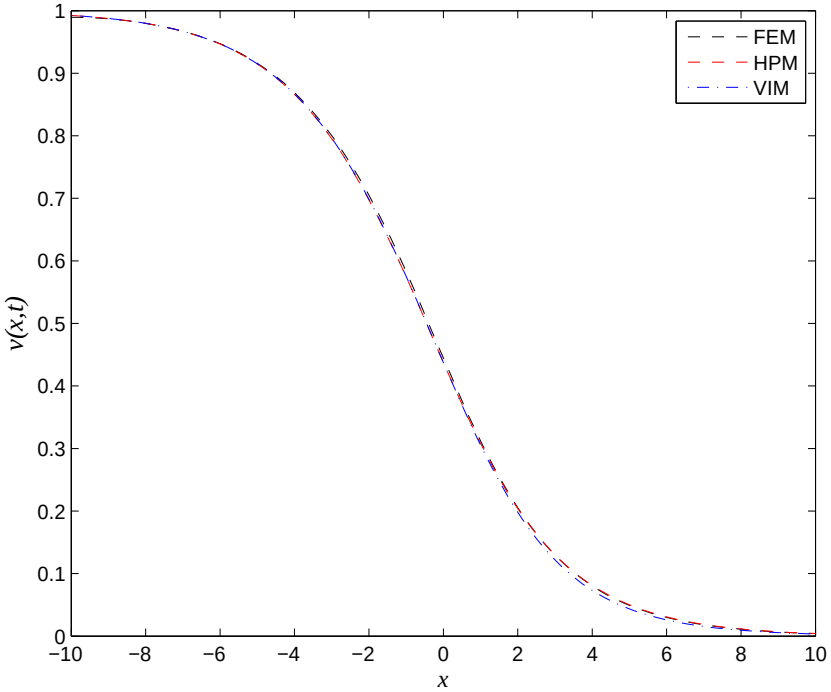


Figure 2: Comparison between the traveling wave solution of $v(x,t)$ using VIM, HPM and COMSOL, when $t = 0.5$.

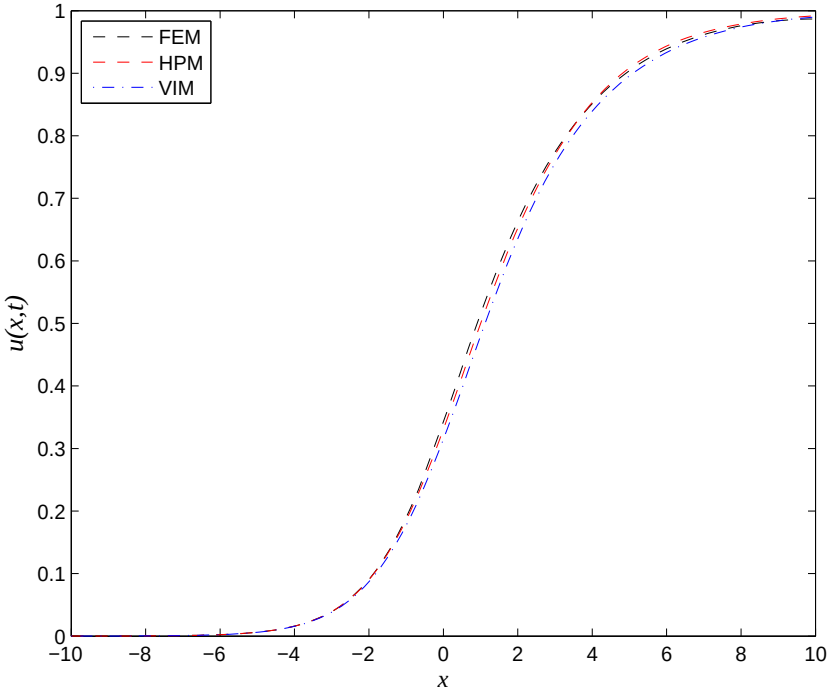


Figure 3: Comparison between the traveling wave solution of $u(x,t)$, using VIM, HPM and COMSOL, when $t = 1$.

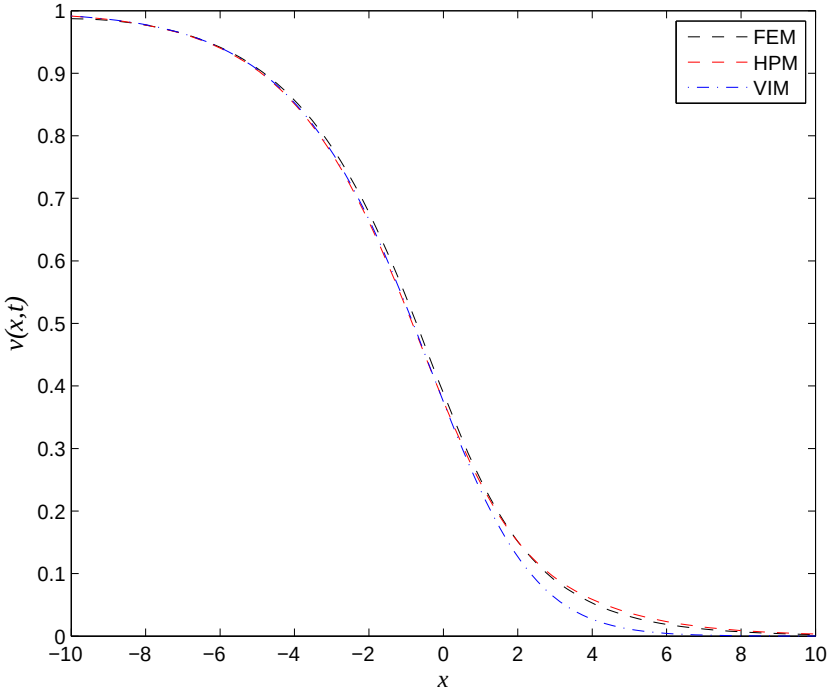


Figure 4: Comparison between the traveling wave solution of $v(x,t)$, using VIM, HPM and COMSOL, when $t = 1$.

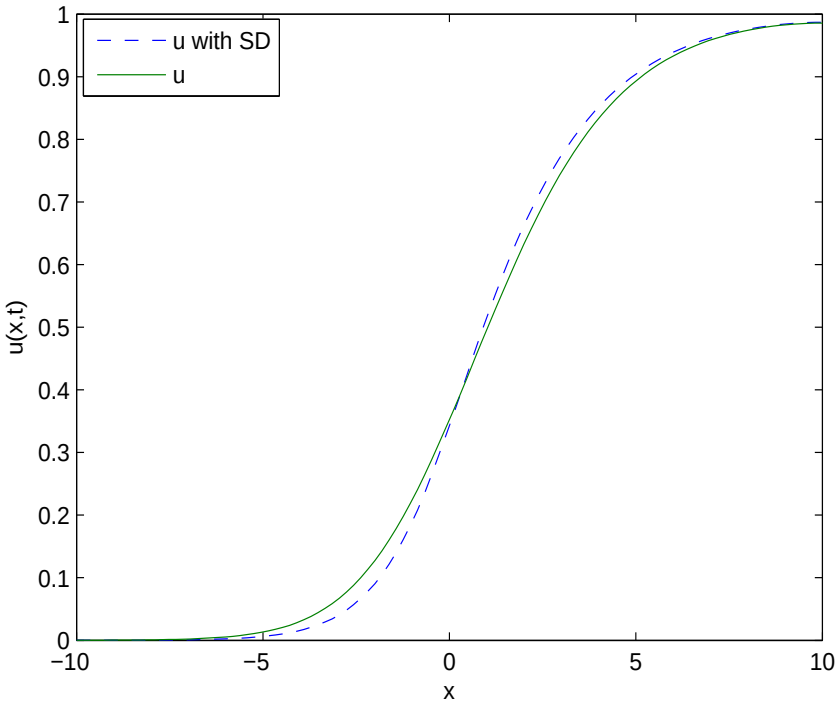


Figure 5: Comparison between traveling wave solution of $u(x,t)$, with and without self-diffusion when $t = 1$ and using COMSOL.

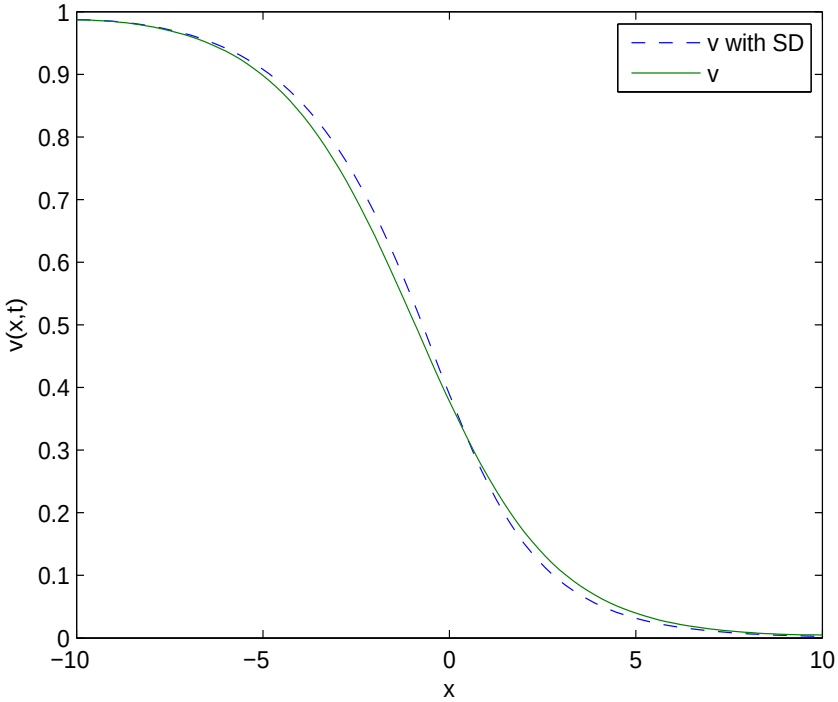


Figure 6: Comparison between traveling wave solution of $v(x,t)$, with and without self-diffusion when $t = 1$ and using COMSOL.