

Evolution operator of the bipartite orbit of Graphs ¹

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Abstract

The operation of substitution consists of replacing a vertex of a graph by another graph.

This new graph is characterized through a function (of substitution) that can be self-definable. The purpose of this work is to construct evolution operators for orbit $\{w^k(G)\}$, where each element of $\{w^k(G)\}$ is obtained by substituting each vertex of the previous element by a graph. Here, both the initial graph G as the family of graphs of substitution, are known. In this paper, simple and finite graphs will be used, framed in the graphs theory's area.

Keywords and Phrases: Graph, distribution operator, substitution of graph, realizable graph, discrete dynamical systems.

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1 Introduction

The graphs to be considered will be simple and finite and with a nonempty set of edges. For a graph G , $V(G)$ denotes the set of vertices and $E(G)$ denotes the set of edges. The cardinality of $V(G)$ is called order of G and often it is written as $|G|$, and the cardinality of $E(G)$ is called size of G and generally it is written as $|E(G)|$. A (p, q) graph has order p and size q . Two vertices u and v are called neighbors if $\{u, v\}$ is an edge of G . For any vertex v of G , denote by N_v the neighbors of v . Other concepts used in this work and not defined explicitly can be found in the references [1], [3], [5], [7], [12], [15], [16].

2 Preliminares

Some essential concepts of this work are the following:

2.1 Substitution

Let suppose that G and K are two graphs disjoint by vertices. For a non isolated vertex v in $V(G)$ and a function $s : N_v \rightarrow V(K)$ it will be defined the substitution of the vertex v by the graph K , as the graph M , denoted by $G(v, s)K$, such that:

- (1) $V(M) = (V(G) \cup V(K)) - \{v\}$ and
- (2) $E(M) = (E(G) \cup E(K) - \{vx/x \in N_v\}) \cup \{xs(x)/x \in N_v\}$.

It is said that the vertex v is the substituted vertex by K in G under the function s and this function is called substitution function. (See [12], [13]).

If v is isolated, then $M = (G - v) \oplus K$.

Definition 1 *Let $V(G) = \{v_1, \dots, v_n\}$ and let $H_1, \dots, H_n$ be a sequence of pairwise vertex disjoint graphs. Iteratively, we define the following graphs: $M_0 = G$ and $M_k = M_{k-1}(v_k, s_k)H_k$ the graph which is obtained by substitution of vertices of G by graphs H_i , $1 \leq i \leq k$.*

In other words, M_1 , denotes a graph obtained by substitution of only one vertex of G , M_2 denotes a graph obtained by substitution of only one vertex of M_1 , and so on. Note that every substituted vertex must belong to $V(G)$.

An edge of the substitution graph M_p is an internal edge if it is $s_i(x)s_i(y)$, [13]. The edge in M_p that not an internal edge is called an external edge [12].

Let G be a graph without isolated vertices, if each vertex v of G is substituted by a complete graph with $val(v)$ vertices, through an injective substitution function, then it will be said that the graph G has been expanded [12]. When each vertex of a given G graph is substituted for a copy of G through injective substitution functions a special type of substitution is obtained which will be called self-substitution and denoted by $G(G)$. See more [13], [14].

2.2 Realizable Graph

A (p, q) graph G is said to be realizable on $\mathbb{R}^3$ if it is possible to distinguish a collection of p different points of $\mathbb{R}^3$, that correspond to the vertices of G and a collection of q curves, pairwise disjoint except, possibly, in extreme points, that correspond to the edges of G such that if a curve γ corresponds to the edge $e = uv$, then only extreme points of γ correspond to vertices of G , namely u and v (More details [3], [5]).

In this work, we use the following realization concept. Here, Ω represents the class of all simple and finite graphs. For a set $A \subset \mathbb{R}^3$ and for $\lambda \in \mathbb{R}^+$ the neighborhood of A is defined, denoted by A_λ , as the subset $\mathbb{R}^3$ defined by

$$A_\lambda = \{y \in \mathbb{R}^3 / \exists x \in A : d(x, y) < \lambda\}.$$

Here d is the usual metric on $\mathbb{R}^3$.

Remark 2 $A_\lambda = \bigcup_{x \in A} V_\lambda(x)$, where $V_\lambda(x) = \{y \in \mathbb{R}^3 / d(x, y) < \lambda\}$

Example 3 $A = \{(1, 2, 1)\} \wedge \lambda = 1 \Rightarrow A_1 = V_1((1, 2, 1))$: Ball of radius 1

Example 4 $A = \{(1, 2, 1), (2, -2, 3)\} \wedge \lambda = \sqrt{2}$, then:

$A_{\sqrt{2}} = V_{\sqrt{2}}((1, 2, 1)) \cup V_{\sqrt{2}}((2, -2, 3))$: Union of two disjoint balls of radius $\sqrt{2}$.

2.3 Realization of a graph:

If $G \in \Omega$ and $h : V(G) \rightarrow \mathbb{R}^3$ is a injective function, then the realization of G in $\mathbb{R}^3$, denoted by G^* , is defined by

$$G^* = \{h(v)/v \in V(G)\} \cup \left(\bigcup_{uv \in E(G)} h(u)h(v) \right),$$

where

$$h(u)h(v) = \{h(u) + t(h(v) - h(u)) / t \in [0, 1]\}.$$

This realization must satisfy the following conditions:

- (i) If $G_1, G_2 \in \Omega$ and $G_1 \neq G_2$, then $G_1^* \cap G_2^* = \phi$.
- (ii) If $G_1, G_2 \in \Omega$ and $G_1^* \cap G_2^* \neq \Phi$, then $G_1 = G_2$.
- (iii) Each $G \in \Omega$ admits one and only one realization G^* . This identification defines the dual class $\Omega^* = \{G^* \subset \mathbb{R}^3 / G \in \Omega\}$.

2.4 Distance on Ω

If $G_1, G_2 \in \Omega$, then the distance from G_1 to G_2 , denoted by $\mathcal{D}(G_1, G_2)$, is defined by $\mathcal{D}(G_1, G_2) = \inf\{\lambda \in \mathbb{R}^+ / G_1^* \subset (G_2^*)_\lambda \wedge G_2^* \subset (G_1^*)_\lambda\}$

Example 5 If $G_1^* = \{i, j, k\} \cup ij \cup ik \cup jk$ and $G_2^* = \{0, \frac{i}{2}\} \cup 0\frac{i}{2}$, then

$$\mathcal{D}(G_1, G_2) = d\left(\frac{i}{2}, j\right) = \frac{\sqrt{5}}{2}$$

Example 6 If $G_1^* = \{0\}$ and $G_2^* = \{j + mi / m \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}\} \cup \left(\bigcup_{t \in \{1, 2, 3, 4\}} \left(j + \frac{(t-1)i}{4}\right) \left(j + \frac{ti}{4}\right)\right)$, then $\mathcal{D}(G_1, G_2) = d(0, j + i) = \sqrt{2}$

2.5 Compactness

If $A \subset \mathbb{R}^3$ and $\varepsilon \in \mathbb{R}^+$, then the compactness of A , denoted by A_ε , is the set $\{y \in \mathbb{R}^3 / \exists x \in A, d(x, y) < \varepsilon\}$. Here, d is usual distance on $\mathbb{R}^3$. Note that $A \subset A_\varepsilon$ and that A_ε is compact according to usual topology of $\mathbb{R}^3$.

For more details *Compactness*, see [15], [16].

2.6 Discrete dynamical systems

A discrete dynamical system is any set X together with a mapping $f : X \rightarrow X$. In this work X is always a set of graphs. In literature [15], X must have some topology on which f is continuous.

For more details *Discrete dynamical systems*, see [8], [9], [15].

2.7 Attractor points

An orbit of f , $f : \Omega \rightarrow \Omega$, being f continuous in topology of Ω , is any sequence of the form $\{G, f(G), \dots, f^n(G), \dots\}$.

Definition 7 *A graph G is an attractor point of f if there exists some natural number $n > 1$ such that $G = f^n(G)$.*

Lemma 8 G^* is compact on $\mathbb{R}^3$.

Proof. See [14]. ■

In the following proposition a distance on Ω will be constructed.

Proposition 9 *The function $\mathbf{D} : \Omega \times \Omega \rightarrow \mathbb{R}$ defined by*

$$\mathbf{D}(G_1, G_2) = \text{Inf}\{\lambda \in \mathbb{R} / G_1^* \subset (G_2^*)_\lambda \wedge G_2^* \subset (G_1^*)_\lambda\}, \text{ is a distance on } \Omega.$$

Proof. See [14]. ■

Theorem 10 $(\Omega, \mathbf{D})$ is a complete metric space.

Proof. See [14]. ■

Now, the following lemma is fundamental.

Lemma 11 *If w is a function of Ω in Ω , defined by $w(G) = M_p(G)$, where p is the order of G , then w is a contraction in Ω .*

Proof. See [14]. ■

The existence of an attractor point to w is assured by the following theorem whose demonstration could be found in [10].

Theorem 12 *If M is a complete metric space and $f : M \rightarrow M$ is a contraction, then there exists $x_f = \lim_{n \rightarrow \infty} f^n(x)$. Moreover, x_f is independent of the choice of x in M . Also x_f is the only fixed point of f .*

Theorem 13 *If $G \in \Omega$, then the orbit $G \rightarrow w(G) \rightarrow \dots w^k(G) \rightarrow \dots$ has a single attractor point G_w for w .*

Proof. Apply Lemma 11 and Theorem 12. ■

See more [1], [16].

3 Evolution Operator

In this Section we introduce the concept of evolution operator and analyze its behavior. A evolution operator of bipartite complete graphs is a pair (γ, λ) where γ is a continuous function (in the topology induced by the distance $\mathbf{D}$ defined in the Theorem 10) of Ω in Ω that choose bipartite complete graph of Ω and λ is a family $\langle s_1, \dots, s_p \rangle$ of functions of substitution relating to each bipartite complete preimage of the function γ such that if G is a bipartite complete graph $K(r, s)$ then $\gamma(K(r, s)) = M_p(K(r, s))$. This operator is constructed through substitution, as it follows:

If $v_1, \dots, v_p$ are the vertices of a bipartite complete graph G and $H_1, \dots, H_p$ is a sequence of graph with no vertices in common between them or with $K(r, s)$ then $M_p(K(r, s))$ will denoted the graph obtained by substitution of p vertices of $K(r, s)$ by graphs H_i , $1 \leq i \leq p$, where $M_0(K(r, s)) = K(r, s)$, $M_1(K(r, s))$, will denoted the graph obtained by substitution of only a vertex of $K(r, s)$ through of a injective function of substitution s_1 , $M_2(K(r, s))$ will denoted the graph obtained by substitution of only a vertex of $M_1(K(r, s))$ through of a injective function of substitution s_2 , and so forth. Note that each substituted vertex must belong to $V(K(r, s))$. Here the family $(s_1, \dots, s_p)$ of substitution functions determine λ and w determine γ , where w is the continuous function built in the Lemma 11. Moreover, if the order of each H_i is p_i , then the order of $M_p(K(r, s))$ is $\sum_{i=1}^p p_i$. Succesively the rest of the elements of the orbit $\{w^k(K(r, s))\}$ are obtained.

Definition 14 The orbit $\{w^k(K(r, s))\}$ is said to be complete bipartite if each substitution function is injective.

In order to illustrate the ideas above then we will give the following examples:

Example 15 Let $G = K(1, 4)$, $V(K(1, 4)) = \{v_1, v_2, v_3, v_4, v_5\}$ with bipartition $A = \{v_1\}$ and $B = \{v_2, v_3, v_4, v_5\}$ with $\lambda = (s_1, \dots, s_5)$, $s_i : N_{v_i} \rightarrow V(H_{deg(v_i)})$, $1 \leq i \leq 5$, injective function. Then $w(K(1, 4)) = M_8(K(1, 4))$

In Figure 1 the operator action is illustred.

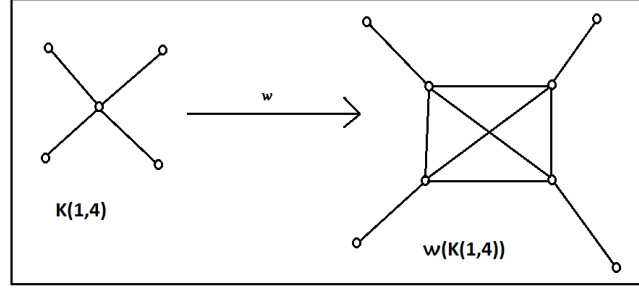


Figure 1: Operator action

Example 16 Let $G = K(2,3)$ and $H_i, 1 \leq i \leq 5$, be a family of complete graphs, disjoint between them and with G , such that if $v \in V(K(2,3))$ then $H_i \equiv K_{deg(v)}$ (where $\equiv$ indicates graphs isomorphism). That way, if $V(K(2,3)) = \{v_1, v_2, v_3, v_4, v_5\}$ with bipartition $A = \{v_1, v_2\}$ and $B = \{v_3, v_4, v_5\}$ with $\lambda = (s_1, \dots, s_5)$, $s_i : N_{v_i} \rightarrow V(H_{deg(v_i)}), 1 \leq i \leq 5$, injective function. Then $w(K(2,3)) = M_{12}(K(2,3))$

In Figure 2 the operator action is illustrated.

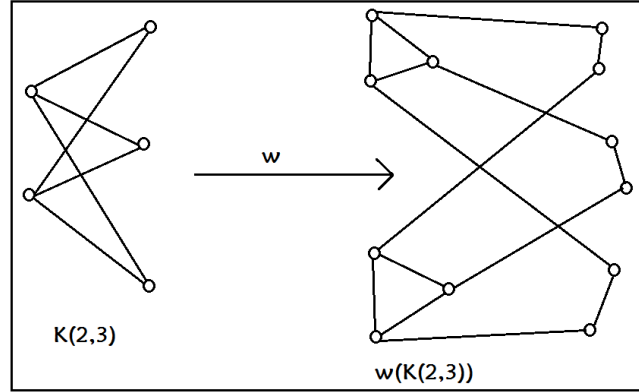


Figure 2: Operator action

The following theorems, in some way, characterize complete bipartite evolution operators.

Theorem 17 If $\{w^k(K(m, p-m))\}, m < p$, is a complete bipartite orbit, then for each k , $n_c(w^k(K(m, p-m))) = \max\{m, p-m\}$.

Proof. Suppose that $n_c(w^k(K(m, p-m))) = \max\{m, p-m\} = m$. By applying the evolution operator to state $w^k(K(m, p-m))$ a complete subgraph

of order m is obtained. As the rest of the blocks of $w^{k+1}(K(m, p-m))$ has order m or $p-m$, we have that $n_c(w^{k+1}(K(m, p-m))) = m$. Analogously if $n_c(w^k(K(m, p-m))) = \max\{m, p-m\} = p-m$. ■

Theorem 18 *If $\{w^k(K(m, p-m))\}$, $m < p$, is a complete bipartite orbit, then for each k ,*

$$d_{HB}(w^k(K(m, p-m))) = \frac{\ln \left(\sum_{v \in V(w^{k-1}(K(m, p-m)))} \deg(v) \right)}{\ln(|w^{k-1}(K(m, p-m))|)},$$

where d_{HB} is the Hausdorff-Besicovitch dimension.

Proof. We have that:

$$|w(K(m, p-m))| = \sum_{v \in V((K(m, p-m)))} \deg(v).$$

in the same way:

$$|w^2(K(m, p-m))| = \sum_{v \in V(w((K(m, p-m))))} \deg(v).$$

Generalizing, we have:

$$|w^k(K(m, p-m))| = \sum_{v \in V(w^{k-1}(K(m, p-m)))} \deg(v).$$

Therefore,

$$d_{HB}(w^k(K(m, p-m))) = \frac{\ln \left(\sum_{v \in V(w^{k-1}(K(m, p-m)))} \deg(v) \right)}{\ln(|w^{k-1}(K(m, p-m))|)}$$

■

4 Conclusions

The present proposal presents a generalization of the concept of substitution [14], allowing the approach of hypotheses in relation to infinite processes [2] and to enrich the asymptotic properties of such discrete dynamical systems [9] and its relationship with the fractal dimensions [15], through the definition of Hausdorff-Besicovitch dimension [11], [4] in this context.

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