

A Semi-numerical Solution To Problem of Flow Between Two Discs

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1 Abstract

The steady flow of a viscous incompressible fluid between two coaxial discs, one rotating and other stationary with suction is analysed. We propose a semi-numerical method in which recurrence relations derived allow to generate large number of universal coefficients in small Reynolds number perturbation series of the solution function. The convergence of the series is restricted by a simple pole using some special techniques the region of validity of the series is increased. The results provided are in excellent agreement with pure numerical studies.

Keywords: Fluid flow between discs, Semi-numerical solution, Pade approximants.

2 Introduction:

The flow of an incompressible fluid between parallel discs, whether porous or non-porous has been a topic of discussion ([1] - [4]) and is of practical interest in many engineering problems. Steawartson and Pearson have considered the flow between two rotating discs.

Many authors considered flow between two discs with uniform suction, injection, while Mellor et al. have considered the flow between stationary and rotating discs. The analytical studies on the problems are [5]. Some careful numerical studies on this problem are those by [6].

The problem considered [7] solved by perturbation series. He could identify pressure gradient and other parameter only for small Reynolds number. Most numerical methods are comparatively tedious and difficult to implement on a computer. For simple geometries, the semi-analytical techniques have advantage over the pure numerical methods. These methods reveal the analytical structure of the solution that is hidden in numerical methods, [8], [9] and his associates successfully exploited the power of these methods. We calculate a sufficient number of universal coefficients of low Reynolds number perturbation series using computer. We have given an accurate analysis of locating the position and prediction nature of singularity. The extraction of singularity yields analytic continuation of the series to high Reynolds number.

3 Formulation of the problem

The governing equations, assuming an axial symmetry are

$$\frac{1}{r} \frac{\partial}{\partial r}(ru_x) + \frac{\partial u_z}{\partial z} = 0 \quad (1)$$

$$u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{u_r}{r} \right) + \frac{\partial^2 u_r}{\partial z^2} \right] \quad (2)$$

$$\frac{u_r}{r} \frac{\partial}{\partial r}(ru_\theta) + u_z \frac{\partial u_\theta}{\partial z} = \nu \left[\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{\partial^2 u_\theta}{\partial z^2} \right] \quad (3)$$

$$u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2} \right] \quad (4)$$

The disk $z = l$ is rotating while, the disc $z = 0$ is stationary. The fluid is extracted with uniform velocity u through the later disc (fig1).

Using the transformations [7]

$$z = l\eta, \quad u_\theta = \Omega r g(\eta), \quad u_z = -u f(\eta), \quad u_r = \frac{u_r}{2l} f'(\eta) \quad (5)$$

Equations (2) to (4) become,

$$f''' + R(f f'' - \frac{1}{2}(f')^2) = C - \lambda R g^2 \quad (6)$$

$$\text{Or } f^{IV} + f f''' + 2\lambda R g g' = 0 \quad (7)$$

$$\frac{1}{R} g'' + f g' - R' g = 0 \quad (8)$$

where,

$$R = \frac{ul}{\nu}, \quad R_1 = \frac{\Omega l^2}{\nu}, \quad \lambda = \frac{2R_1^2}{R^2} \quad \text{and } C \text{ is constant of integration.}$$

The boundary conditions are

$$f'(0) = 0, \quad f'(1) = 0, \quad g(0) = 0, \quad g(1) = 1, \quad f(0) = 1, \quad f(1) = 0 \quad (9)$$

4 Method of solution

The solution for equations(7) and (8) can be expressed for small values of R as,

$$f = f_0 + \sum_{n=1}^{\infty} R^n f_n(\eta), \quad g = g_0 + \sum_{n=1}^{\infty} R^n g_n(\eta) \quad (10)$$

Equating like powers of R on both sides, we get

$$f_n^{iv} = - \sum_{r=1}^n f_{r-1} f_{n-r}''' - 2\lambda \sum_{r=1}^n g_{r-1} g_{n-r}' \quad (11)$$

$$g_n'' = - \sum_{r=1}^n f_{r-1} g_{n-r}' + \sum_{r=1}^n f_{r-1}' g_{n-r} \quad (12)$$

$$n = 0, 1, 2, \dots$$

The relevant boundary conditions are

$$f'_n(0) = 0 \quad , f'_n(1) = 0$$

$$g_n(0) = 0, \quad g_n(1) = 0 \quad (13)$$

$$f_n(0) = 0, \quad f_n(1) = 0 \quad \text{for } n > 0.$$

The solution of (11) and (12) are

$$f_0(\eta) = (2\eta^3 - 3\eta^2 + 1)$$

$$f_1(\eta) = \frac{1}{420}[-12\eta^7 - 42\eta^6 - 7\lambda\eta^5 + 210\eta^4 + (21\lambda - 192)\eta^3 + (36 - 14\lambda)\eta^2] \quad (14)$$

$$\begin{aligned} f_2(\eta) = & \frac{1}{641}\eta^{11} - \frac{3}{350}\eta^{10} + (6 - \frac{\lambda}{30})\frac{\eta^9}{504} + (\frac{36}{5} - \frac{\lambda}{10})\frac{\eta^8}{336} - (\frac{3\lambda}{10} + \frac{552}{35})\frac{\eta^7}{10} + (\frac{188}{35} + \frac{53\lambda}{30})\frac{\eta^6}{120} \\ & + (6 - \frac{11\lambda}{60})\frac{\eta^5}{60} - (\frac{156}{35} + \frac{3\lambda}{10})\eta^4 + (\frac{31469}{2695} + \frac{779\lambda}{315})\frac{\eta^3}{60} - (\frac{289}{5040} + \frac{8073}{26950})\frac{\eta^2}{2} \end{aligned}$$

and

$$g_0 = \eta$$

$$g_1 = -\frac{1}{20}[4\eta^5 - 5\eta^4 - 10\eta^2 + 11\eta] \quad (15)$$

$$g_2 = \frac{1}{25200} \left[-340\eta^9 + 1260\eta^8 - (40\lambda + 900)\eta^7 + 2940\eta^6 + (5760 + 126\lambda)\eta^5 - (70\lambda + 4125)\eta^4 + 4200\eta^3 - 6930\eta^2 + (4015 - 16\lambda)\eta \right]$$

It is very much necessary to get higher approximation in the series if it has to reveal the true nature of the function represented by it, as we go for higher approximations the algebra involved becomes cumbersome. So, we propose a systematic series expansion scheme which is quite useful and efficient in the calculation. Towards this, we consider f_n , g_n in the form,

$$f_n = \sum_{k=1}^{4n+1} C_{n,k}(1-\eta)\eta^{k+1}, \quad g_n = \sum_{k=1}^{4n} D_{n,k}(1-\eta)\eta^k \quad (16)$$

$n = 1, 2, 3, \dots$

We substitute expression (16) in (11) and (12) and equate various powers of η on both sides till we get general recurrence relation.

$$C_{n(k)} = 2C_{n(k+1)} - C_{n(k+2)} - \frac{1}{(k+3)(k+2)(k+1)k} \left[\sum_{i=1}^2 C_{(n-1)(k-i)}, Q_i(k-i) \right. \\ \left. + \sum_{r=1}^{n-2} \sum_{i=0}^4 \sum_{s=1}^{nk'+i} C_{r(l-nk'-i)} C_{(m-1)(t-3)} Q_{6+i}(t-3) \right. \\ \left. + \sum_{r=1}^{n-2} \sum_{i=0}^2 \sum_{j=1}^{nk'+2+i} D_{r(l-nk'-i)} D_{(m-1)(t-2)} Q_{3+i}(l-nk'-l-i) \right] \quad (17)$$

and

$$D_{n,k} = D_{n,(k+1)} - \frac{1}{(k+1)k} \left[\sum_{i=1}^3 C_{n-1(k-1-i)} P_{i(k-1-i)} \right. \\ \left. + \sum_{r=1}^{n-2} \sum_{i=1}^3 \sum_{j=1}^{nk+i} C_{r(l-nk-i)} D_{(m-1)(t-2)} P_{(4+i)(l-nk-i,t-2)} \right] \quad (18)$$

where, $nk = 4n - 1 - k$, $m = n - r$, $l = 3r + j - 1$, $t = 3m - j$, $k = 1, 2, \dots, 4n + 1$

where, $nk' = 4n - 3 - k$, $m = n - r$, $l = 3r + j - 1$, $t = 3m - j$, $k = 1, 2, \dots, 4n$

where P_i , Q_i are functions of k , k_1 and λ

The expressions for

$$f'(\eta) = 6\eta^2 - 6\eta + \sum_{n=1}^{\infty} R^n C_{n,k} [(k+1)\eta^k - (k+2)\eta^{k+1}] \quad (19)$$

$$g'(\eta) = 1 + \sum_{n=1}^{\infty} R^n D_{n,k} [k\eta^{k-1} - (k+1)\eta^k] \quad (20)$$

The pressure gradient [7]

$$C = f'''(0) + Rf''(0) \quad (21)$$

5 Results and Discussion

A new type of series method is used for studying the problem of flow between two discs using recurrences (17) and (18). In this process we generate the universal coefficients $C_{n,k}$, $D_{n,k}$ $n = 1, 2, 3, \dots, 25$, $k = 4n + 1$, $k = 4n$ respectively. These coefficients in turns give coefficients of the series (21). We locate the singularity of the series using Domb-sykes plot (fig2).

The series (21) summed using pade approximants. Earlier numerical results were only up to $R = 1$, we were able to go up to $R = 50$ using Pade approximants. These results are in close agreement with numerical findings of [5] which are shown in Tables 1 and 2. The results of velocity profile are in good agreement with numerical findings (fig3). For simple geometry the method used here, provides accurate results and has advantages over pure numerical results. In numerical methods a separate scheme is to be developed for calculating derivatives. This is an elaborate process and numerical methods are sensitive to grid size. However, this difficulty is not there in the case of the present method.

Table 1

Comparison of results		
R	Pressure gradient $\lambda =$ 0 Phan-Thien	Series method
1	0.2992	0.2992
5	1.41101	1.411
10	2.4100	2.41
15	2.941537	2.9425
18	3.086	3.06
25	-	3.02
40	-	2.98
50	-	2.91

Table 2

Results for pressure gradient.		
λ	R	C
2	1	5.25
	10	4.21
	40	3.91
3	1	4.14
	10	3.21
	40	3.001
5	1	3.94
	10	3.21
	40	2.91

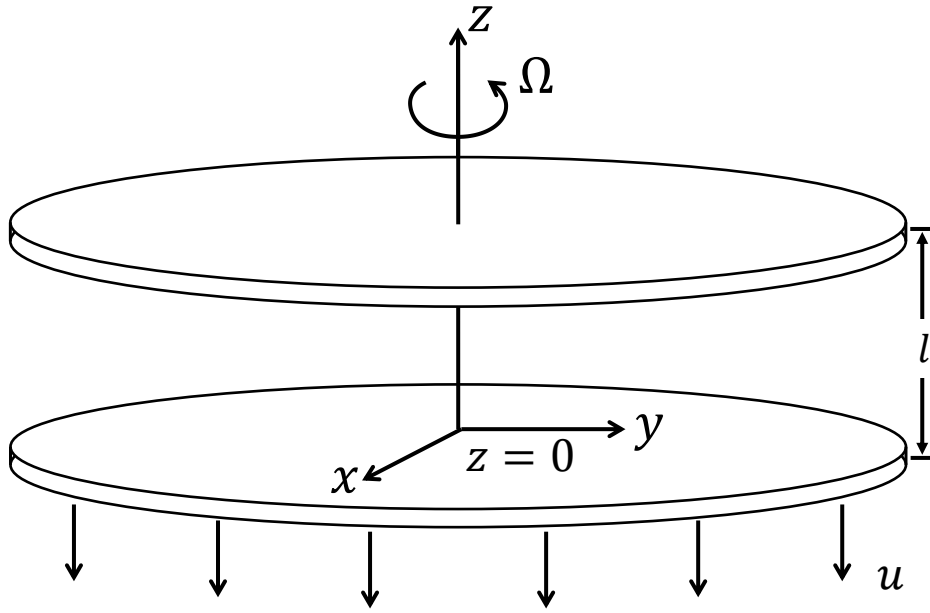


Fig1 : Schematic diagram

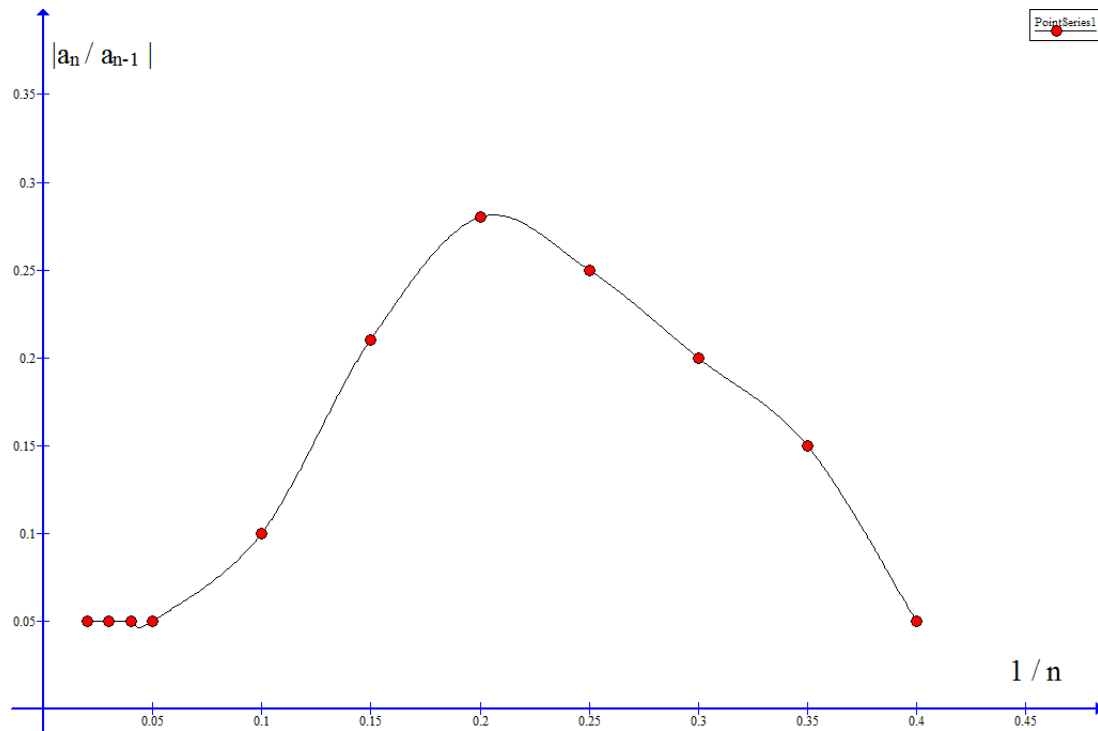


Fig2 : Domb-Sykes Plot for the coefficients

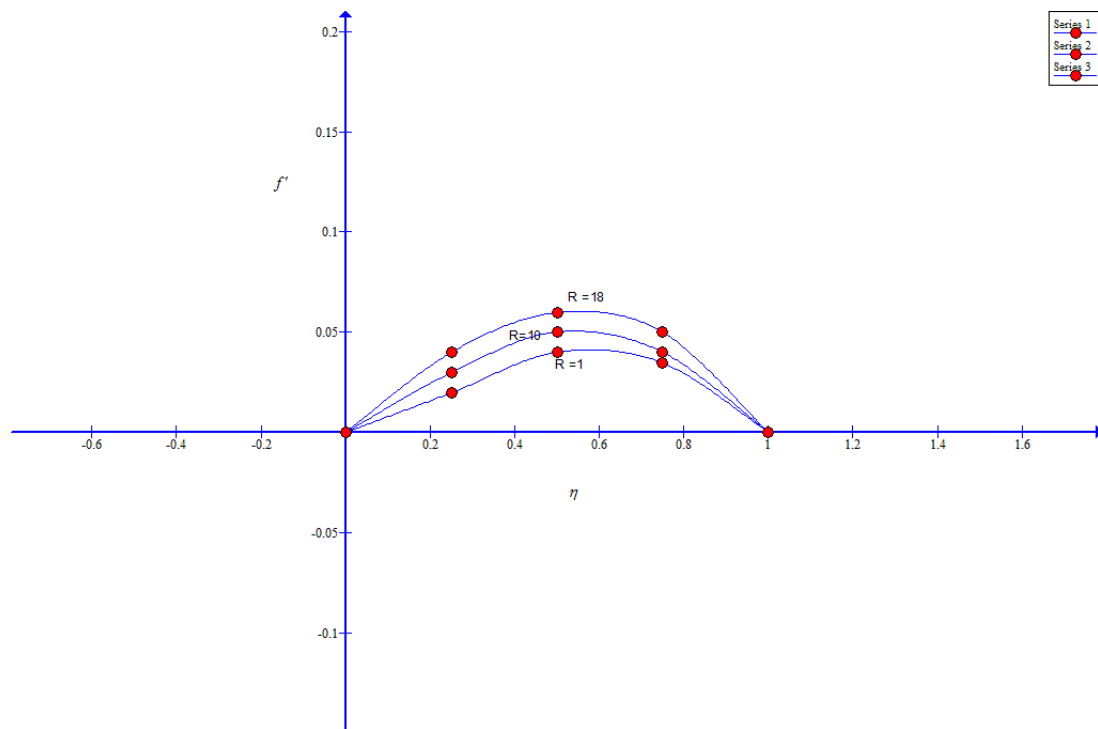


Fig3 : Velocity profile

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