

V_4 Magic Labelings of some graphs

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Abstract

Let A be an abelian group with identity element 0. A graph $G = (V, E)$ is said to admit an a -sum A -magic labeling if there exists an edge labeling $\ell : E(G) \rightarrow A \setminus \{0\}$ and $a \in A$ such that the induced vertex labeling $\ell^+ : V(G) \rightarrow A$ defined by

$$\ell^+(u) = \sum \{\ell(uv) : uv \in E(G)\}$$

is the constant map, $\ell^+(u) = a$ for all $u \in V(G)$. If $a = 0$, the labeling ℓ is called a zero-sum A -magic labeling of G . A graph G is said to be a -sum (resp.zero-sum) A -magic if G admits an a -sum (resp.zero-sum) A -magic labeling. In this paper we will consider the Klein 4 group $V_4 = \{0, a, b, c\} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and investigate graphs that are a -sum A -magic, zero-sum A -magic and both a -sum and zero-sum A -magic.

Keywords: V_4 magic graph; a -sum V_4 magic graph; zero-sum V_4 magic graph

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1 Introduction

In this paper all graphs are connected, finite, simple and undirected. For graph theoretic notations and terminology not directly defined in this paper, we refer to readers [Chartrand & Zhang (2005)].

For an abelian group A , written additively, any mapping $\ell : E(G) \rightarrow A \setminus \{0\}$ is called a labeling, where 0 denote the identity element in A . For any abelian group A , a graph $G = (V, E)$ is said to be A -magic if there exists a labeling $\ell : E(G) \rightarrow A \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \rightarrow A$ defined by

$$\ell^+(u) = \sum \{\ell(uv) : uv \in E(G)\}$$

is a constant map [Lee et al. (2002)]. Observe that A -magic labeling of a graph need not be unique. The V_4 magic graphs was first introduced by S. M. Lee et al. in 2002 [Lee et al. (2002)]. There has been an increasing interest in the study of V_4 magic graphs since the publication of [Lee et al. (2002)].

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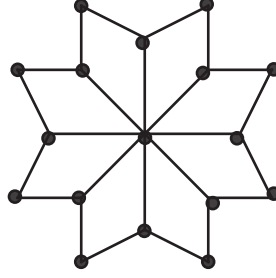


Figure 1: The Jahangir graph $J_{2,8}$

We follow the following definitions and notations described in our earlier publications [Anil Kumar & Vandana(2015);Vandana & Anil Kumar(2015)]. A V_4 magic graph G is called a -sum V_4 magic labeling of G , if there exists a labeling $\ell : E \rightarrow V_4 \setminus \{0\}$ such that $\ell^+(v) = a$ for all $v \in V$. Any graph that admits an a -sum V_4 magic labeling is called an a -sum V_4 magic graph. When $a = 0$, we call G a zero-sum V_4 magic graph.

- (i) $\mathcal{V}_a$, the class of a -sum V_4 magic graphs,
- (ii) $\mathcal{V}_0$, the class of zero-sum V_4 magic graphs, and
- (iii) $\mathcal{V}_{a,0}$, the class of graphs which are both a -sum and zero -sum V_4 magic.

In this paper, we investigate a class of graphs that belongs to the above categories.

2 Main Theorems

Definition 2.1. The Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph consisting of a cycle C_{nm} with one additional vertex called the central vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm} .

Observe that $J_{n,m}$ has $nm + 1$ vertices. The Jahangir graph $J(2,8)$ is shown in Figure 1.

Lemma 2.1. If $\ell : E(J_{n,m}) \rightarrow V_4 \setminus \{0\}$ is an a -sum V_4 magic labeling of $J_{n,m}$, then

$$\sum_{i=1}^m \ell^+(u_i) + \sum_{i=1}^m \sum_{j=1}^{n-1} \ell^+(v_{ij}) + \ell^+(w) = 0$$

where $u_1, u_2, \dots, u_m$ are the m vertices of C_{nm} which is adjacent to the central vertex w and $v_{i1}, v_{i2}, \dots, v_{i(n-1)}$ are the $(n-1)$ vertices between u_i and u_{i+1} , $i = 1, 2, \dots, m$ where $u_{m+1} = u_1$.

Proof. We have

$$\ell^+(w) = \sum_{i=1}^m \ell(u_i w). \quad (1)$$

For $i = 1, 2, \dots, m$, we have

$$\ell^+(u_i) = \ell(u_i w) + \ell(u_i v_{i1}) + \ell(u_i v_{(i-1)(n-1)}), \quad (2)$$

$$\ell^+(v_{ij}) = \ell(v_{ij} v_{i(j+1)}) + \ell(v_{i(j-1)} v_{ij}), j = 2, 3, \dots, n-2, \quad (3)$$

$$\ell^+(v_{i1}) = \ell(u_i v_{i1}) + \ell(v_{i1} v_{i2}), \quad (4)$$

$$\ell^+(v_{i(n-1)}) = \ell(v_{i(n-1)} u_{i+1}) + \ell(v_{i(n-1)} v_{i(n-2)}). \quad (5)$$

From equations (2),(4),(5) we get

$$\sum_{i=1}^m \ell^+(u_i) + \sum_{i=1}^m \ell^+(v_{i1}) + \sum_{i=1}^m \ell^+(v_{i(n-1)}) = \ell^+(w) + \sum_{i=1}^m \ell(v_{i1} v_{i2}) + \sum_{i=1}^m \ell(v_{i(n-1)} v_{i(n-2)}). \quad (6)$$

From equation (3) we get,

$$\sum_{i=1}^m \sum_{j=2}^{n-2} \ell^+(v_{ij}) = \sum_{i=1}^m \sum_{j=2}^{n-2} \ell(v_{ij} v_{i(j+1)}) + \sum_{i=1}^m \sum_{j=2}^{n-2} \ell(v_{i(j-1)} v_{ij}). \quad (7)$$

Adding the equations (6) and (7) we obtain,

$$\sum_{i=1}^m \ell^+(u_i) + \sum_{i=1}^m \sum_{j=1}^{n-1} \ell^+(v_{ij}) = \ell^+(w).$$

This implies that,

$$\sum_{i=1}^m \ell^+(u_i) + \sum_{i=1}^m \sum_{j=1}^{n-1} \ell^+(v_{ij}) + \ell^+(w) = 0.$$

This completes the proof of the lemma. $\square$

Theorem 2.2. $J_{n,m} \in \mathcal{V}_a$ if and only if both n and m are odd.

Proof. Let the vertices of $J_{n,m}$ be as in the proof of Lemma 2.1. Also let $u_i = u_{i(mod m)}$ and $v_{ij} = v_{i(mod m)j(mod n)}$. First assume that $J_{n,m} \in \mathcal{V}_a$. Then we have $(nm+1)a = 0$. This implies that $nm+1$ is even which in turn implies that both n and m are odd. Conversely, assume that n and m are odd. Define a labeling $\ell : E(J_{n,m}) \rightarrow V_4 \setminus \{0\}$ as follows.

For $i = 1, 2, \dots, m$ do :

$$\ell(u_i w) = a,$$

$$\ell(u_i v_{i1}) = b,$$

end for

$$\ell(u_1 v_{m(n-1)}) = b,$$

$$\ell(u_i v_{(i-1)(n-1)}) = b, i = 2, 3, \dots, m$$

For $i = 1, 2, \dots, m$ do :

$$\ell(v_{ij} v_{i(j+1)}) = c, j = 1, 3, \dots, n-2,$$

$$\ell(v_{ij} v_{i(j+1)}) = b, j = 2, 4, \dots, n-3.$$

end for

With this labeling we get

$$\begin{aligned}\ell^+(w) &= ma = a, \\ \ell^+(u_i) &= b + b + a = a, \\ \ell^+(v_{ij}) &= b + c = a.\end{aligned}$$

Obviously, ℓ is an a -sum V_4 magic labeling of $J_{n,m}$. □

Theorem 2.3. $J_{n,m} \in \mathcal{V}_0$ for all n and m .

Proof. Let the vertices of $J_{n,m}$ be as in the proof of Theorem 2.2. We consider the following cases.

Case 1: m even

Define a labeling $\ell : E(J_{n,m}) \rightarrow V_4 \setminus \{0\}$ as follows.

$$\begin{aligned}\ell(u_i w) &= c, \text{ for } i = 1, 2, \dots, m, \\ \text{For } j = 1, 2, \dots, n-2 \text{ do :} \\ \ell(v_{ij} v_{i(j+1)}) &= a, i = 1, 3, \dots, m-1, \\ \ell(v_{ij} v_{i(j+1)}) &= b, i = 2, 4, \dots, m. \\ \text{end for} \\ \ell(u_i v_{i1}) &= a, \text{ for } i = 1, 3, \dots, m-1, \\ \ell(u_i v_{i1}) &= b, \text{ for } i = 2, 4, \dots, m, \\ \ell(u_{i+1} v_{i(n-1)}) &= \begin{cases} a, & i = 1, 3, \dots, m-1 \\ b, & i = 2, 4, \dots, m+1 \end{cases}\end{aligned}$$

Obviously, ℓ is a 0-sum V_4 magic labeling of $J_{n,m}$.

Case 2: m odd

Define a labeling $\ell : E(J_{n,m}) \rightarrow V_4 \setminus \{0\}$ as follows.

$$\begin{aligned}\ell(u_i w) &= a, \text{ for } i = 1, 4, 5, \dots, m, \\ \ell(u_2 w) &= b, \\ \ell(u_3 w) &= c, \\ \text{For } j = 1, 2, \dots, n-2 \text{ do :} \\ \ell(v_{2j} v_{2(j+1)}) &= a, \\ \ell(v_{ij} v_{i(j+1)}) &= b, \text{ for } i = 3, 5, \dots, m, \\ \ell(v_{ij} v_{i(j+1)}) &= c, \text{ for } i = 4, 6, \dots, m-1, m+1. \\ \text{end for} \\ \ell(u_2 v_{21}) &= \ell(u_3 v_{2(n-1)}) = a, \\ \ell(u_i v_{i1}) &= \ell(u_{i+1} v_{i(n-1)}) = b, \text{ for } i = 3, 5, \dots, m, \\ \ell(u_i v_{i1}) &= \ell(u_{i+1} v_{i(n-1)}) = c, \text{ for } i = 4, 6, \dots, m-1, m+1.\end{aligned}$$

Obviously, ℓ is a 0-sum V_4 magic labeling of $J_{n,m}$. □

Theorem 2.4. If both m and n are odd, $J_{n,m} \in \mathcal{V}_{a,0}$.

Proof. The proof follows from Theorems 2.2 and 2.3. □

Definition 2.2. The windmill graph $D_n^{(m)}$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common.

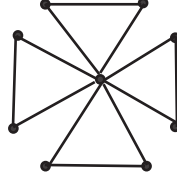


Figure 2: Windmill graph $D_3^{(4)}$

The windmill graph $D_3^{(4)}$ is shown in Figure 2. The graph $D_3^{(m)}$ is called the Dutch windmill graph or the friendship graph, F_m .

Lemma 2.5. *If $\ell : E(D_n^{(m)}) \rightarrow V_4 \setminus \{0\}$ is an a -sum V_4 magic labeling of $D_n^{(m)}$, then*

$$\sum_{i=1}^m \sum_{j=1}^{n-1} \ell^+(u_j^i) = \ell^+(v)$$

where $u_1^i, u_2^i, \dots, u_{n-1}^i$ are the vertices of i^{th} copy of K_n in $D_n^{(m)}$ and v is the common vertex.

Proof. The proof is similar to Lemma 2.1 . □

Theorem 2.6. *The windmill graph $D_n^{(m)} \in \mathcal{V}_a$ if and only if m is odd and n is even.*

Proof. Suppose $D_n^{(m)} \in \mathcal{V}_a$. Then by Lemma 2.5, $[m(n-1)+1]a = 0$. This implies that $m(n-1)$ is odd. This holds only when m is odd and n is even. Conversely suppose that m is odd and n is even. Let the vertices of $D_n^{(m)}$ be as in the Lemma 2.5. Define a labeling $\ell : E(D_n^{(m)}) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} &\text{For } i = 1, 2, \dots, m \text{ do :} \\ &\quad \ell(u_j^i v) = a, j = 1, 2, \dots, n, \\ &\quad \ell(u_j^i u_{j+1}^i) = a, j = 1, 2, \dots, n. \\ &\text{end for} \end{aligned}$$

Obviously ℓ is an a -sum V_4 magic labeling of $D_n^{(m)}$. □

Theorem 2.7. *$D_n^{(m)} \in \mathcal{V}_0$ for all n and m .*

Proof. We consider the following cases.

Case 1: n is odd.

Label all the edges by a . Then we have $\ell^+(v) = 0$ for all $v \in V(D_n^{(m)})$.

Case 2: n is even.

Define a labeling $\ell : E(D_n^{(m)}) \rightarrow V_4 \setminus \{0\}$ by

For $i = 1, 2, \dots, m$ do :
 $\ell(u_j^i u_{j+1}^i) = a, j = 1, 3, \dots, n-1,$
 $\ell(u_j^i u_{j+1}^i) = b, j = 2, 4, \dots, n,$
 $\ell(u_j^i u_k^i) = c, j, k = 1, 2, \dots, n, k \neq j+1.$
 end for

Thus we get $\ell^+(v) = 0$ for all $v \in V(D_n^{(m)})$. Obviously ℓ is a zero-sum V_4 magic labeling of $D_n^{(m)}$. This completes the proof of the theorem. □

Theorem 2.8. $D_n^{(m)} \in \mathcal{V}_{a,0}$ if and only if m is odd and n is even.

Proof. The proof follows from theorems 2.6 and 2.7. □

Theorem 2.9. $F_m \notin \mathcal{V}_a$ for any m .

Proof. Observe that F_m is the one-point union of m copies of a rooted triangle. Let the vertices of the i^{th} copy be $0, u_i$ and v_i . Assume that 0 is the root of the triangles. If F_m admits an a -sum V_4 magic labeling, then

$$\ell^+(u_i) = \ell^+(v_i) = a.$$

This implies that for all i ,

$$\begin{aligned} \ell(u_i v_i) = b, \ell(0 u_i) = \ell(0 v_i) = c \\ \text{or } \ell(u_i v_i) = c, \ell(0 u_i) = \ell(0 v_i) = b. \end{aligned}$$

In both the cases, $\ell^+(0) = 2ma = 0$. This is a contradiction. □

Theorem 2.10. $F_m \in \mathcal{V}_0$ for all m .

Proof. Label all the edges by a . Obviously this is a zero-sum V_4 magic labeling of F_m . □

Theorem 2.11. $F_m \notin \mathcal{V}_{a,0}$ for any m .

Proof. The proof follows from theorems 2.9 and 2.10. □

Definition 2.3 (Sweetly & Paulraj Joseph (2010)). The graph $P_2 \square P_n$ is called a ladder. It is denoted by L_n .

Theorem 2.12. Ladders L_n are a -sum V_4 magic for all n .

Proof. Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ be the vertices of a ladder L_n such that $E(G) = \{u_i u_{(i+1)}/i = 1, 2, \dots, n-1\} \cup \{v_j v_{(j+1)}/j = 1, 2, \dots, n-1\} \cup \{u_i v_i/i = 1, 2, \dots, n\}$. Define a labeling $\ell : E(L_n) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_1 v_1) = \ell(u_n v_n) = b, \\ \ell(u_i v_i) = a, \text{ for } i = 2, 3, \dots, n-1, \\ \ell(u_i u_{(i+1)}) = \ell(v_i v_{(i+1)}) = c, \text{ for } i = 1, 2, \dots, n-1. \end{aligned}$$

Then clearly ℓ is an a -sum V_4 magic labeling of L_n . □

Theorem 2.13. [Sweetly & Paulraj Joseph (2010)] $L_n \in \mathcal{V}_0$ for all n .

Theorem 2.14. $L_n \in \mathcal{V}_{a,0}$ for all n .

Proof. The proof follows from theorems 2.12 and 2.13. □

Definition 2.4 (Sweetly & Paulraj Joseph (2010)). The graph G with the vertex set $\{u_0, u_1, \dots, u_{n+1}, v_0, v_1, \dots, v_{n+1}\}$ and the edge set $\{u_i u_{i+1}, v_i v_{i+1} : 0 \leq i \leq n\} \cup \{u_i v_i : i = 1, 2, \dots, n\}$ is called ladder L_{n+2} .

Theorem 2.15 (Sweetly & Paulraj Joseph (2010)). $L_{n+2} \in \mathcal{V}_a$ for all n .

Theorem 2.16. $L_{n+2} \notin \mathcal{V}_0$ for any n .

Proof. Since the graph has pendant edges it is not zero-sum V_4 magic for any n . □

Definition 2.5 (Sweetly & Paulraj Joseph (2010)). The graph G with the vertex set $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and edge set $\{u_i u_{(i+1)}, v_i v_{(i+1)}, v_i u_{(i+1)} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$ is called a semiladder of length n .

Theorem 2.17. Semiladders are a -sum V_4 magic for all n .

Proof. Let G be a semiladder of length n . We consider two cases.

Case 1: n odd

Define a labeling $\ell : E(G) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
 \ell(u_1 v_1) &= b, \\
 \ell(u_i v_i) &= a, \text{ for } i = 2, 3, \dots, n-1, \\
 \ell(u_n v_n) &= b, \\
 \ell(v_i u_{(i+1)}) &= a, \text{ for } i = 1, 2, \dots, n-1. \\
 &\text{For } i = 1, 3, \dots, n-2 \text{ do :} \\
 \ell(u_i u_{(i+1)}) &= c, \\
 \ell(v_i v_{(i+1)}) &= b. \\
 &\text{end for} \\
 &\text{For } i = 2, 4, \dots, n-1 \text{ do :} \\
 \ell(u_i u_{(i+1)}) &= b, \\
 \ell(v_i v_{(i+1)}) &= c. \\
 &\text{end for}
 \end{aligned}$$

Thus ℓ is an a -sum V_4 magic labeling of G .

Case 2: n even

Define a labeling $\ell : E(G) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
&\ell(u_1v_1) = b, \\
&\ell(u_iv_i) = a, \text{ for } i = 2, 3, \dots, n-1, \\
&\ell(u_nv_n) = c, \\
&\ell(v_iu_{i+1}) = a, \text{ for } i = 1, 2, \dots, n-1, \\
&\quad \text{For } i = 1, 3, \dots, n-1 \text{ do :} \\
&\quad \ell(u_iu_{i+1}) = c, \\
&\quad \ell(v_iv_{i+1}) = b. \\
&\quad \text{end for} \\
&\quad \text{For } i = 2, 4, \dots, n-2 \text{ do :} \\
&\quad \ell(u_iu_{i+1}) = b, \\
&\quad \ell(v_iv_{i+1}) = c. \\
&\quad \text{end for}
\end{aligned}$$

Thus ℓ is an a -sum V_4 magic labeling of G . □

Theorem 2.18. [Sweetly & Paulraj Joseph (2010)] Semiladders are zero-sum V_4 magic for all n .

Theorem 2.19. If G is a semiladder, then $G \in \mathcal{V}_{a,0}$.

Proof. The proof follows from theorems 2.17 and 2.18. □

Definition 2.6 (Sweetly & Paulraj Joseph (2010)). Composition of two graphs $G[H]$ has $V(G) \times V(H)$ as vertex set in which (g_1, h_1) is adjacent to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$.

The graph $P_4[K_2^c]$ is shown in Figure 3.

Theorem 2.20. The composition $P_n[K_2^c]$ is a -sum V_4 magic for all n .

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of P_n and x, y be that of K_2^c . Let u_i denote the vertices (v_i, x) and w_i denote (v_i, y) of $P_n[K_2^c]$, $1 \leq i \leq n$. Define a labeling $\ell : E(P_n[K_2^c]) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
&\ell(u_iu_{i+1}) = b, \text{ for } i = 1, 2, \dots, n-1, \\
&\ell(w_1w_2) = b, \\
&\ell(w_iw_{i+1}) = c, \text{ for } i = 2, 3, \dots, n-1, \\
&\ell(u_1w_2) = c, \\
&\ell(u_iw_{i+1}) = b, \text{ for } i = 2, 3, \dots, n-1, \\
&\ell(u_{i+1}w_i) = c, \text{ for } i = 1, 2, \dots, n-1.
\end{aligned}$$

Thus ℓ is an a -sum V_4 magic labeling of $P_n[K_2^c]$. □

Theorem 2.21. [Sweetly & Paulraj Joseph (2010)] $P_n[K_2^c] \in \mathcal{V}_0$ for all n .

Theorem 2.22. $P_n[K_2^c] \in \mathcal{V}_{a,0}$ for all n .

Proof. The proof follows from theorems 2.20 and 2.21. □

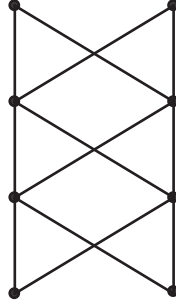


Figure 3: $P_4[K_2^c]$

Definition 2.7 (Sweetly & Paulraj Joseph (2010)). One point union of any number of connected graphs is obtained by identifying one vertex from each graph. One point union of t cycles each of length n is denoted by $C_n^{(t)}$.

Lemma 2.23. If $\ell : C_n^{(t)} \rightarrow V_4 \setminus \{0\}$ is an a -sum magic labeling of $C_n^{(t)}$, then

$$\sum_{i=1}^{n-1} \sum_{j=1}^t \ell^+(u_{ij}) = \ell^+(v)$$

where $u_{1j}, u_{2j}, \dots, u_{(n-1)j}$ are the vertices of j -th copy of C_n and v is the common vertex.

Proof. The proof is similar to Lemma 2.1. □

Theorem 2.24. $C_n^{(t)} \in \mathcal{V}_a$ if and only if n is even and t is odd.

Proof. First assume that $C_n^{(t)} \in \mathcal{V}_a$. Then by Lemma 2.23, $[(n-1)t+1]a = 0$. This equation holds if and only if n is even and t is odd. Conversely suppose that n is even and t is odd. Define a labeling $\ell : C_n^{(t)} \rightarrow V_4 \setminus \{0\}$ as follows

For $j = 1, 2, \dots, t$ do :
 $\ell(u_{ij}u_{(i+1)j}) = b$, for $i = 1, 3, \dots, n-1$,
 $\ell(u_{ij}u_{(i+1)j}) = c$, for $i = 2, 4, \dots, n$.
end for

Obviously $\ell^+(u_{ij}) = a$, $\ell^+(v) = a$. □

Theorem 2.25. [Sweetly & Paulraj Joseph (2010)] $C_n^{(t)} \in \mathcal{V}_0$ for all n and t .

Theorem 2.26. $C_n^{(t)} \in \mathcal{V}_{a,0}$ if and only if n is even and t is odd.

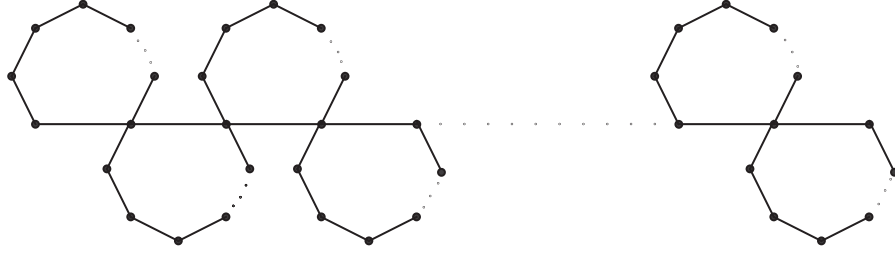


Figure 4: Snake Graph $T_n^{(m)}$

Proof. The proof follows from theorems 2.24 and 2.25. $\square$

Definition 2.8 (Sweetly & Paulraj Joseph (2010)). A snake graph is formed by taking n -copies of a cycle C_m and identifying exactly one edge of each copy to a distinct edge of the path $P_{(n+1)}$, which is called as the backbone of the snake. It is denoted by $T_n^{(m)}$.

The graph $T_n^{(m)}$ is shown in Figure 4.

Theorem 2.27. $T_n^{(m)} \in \mathcal{V}_a$ if and only if m is even and n is odd.

Proof. Suppose that $T_n^{(m)} \in \mathcal{V}_a$. Let $u_{ij}, i = 1, 2, \dots, n, j = 1, 2, \dots, m$ be the vertices in the graph. Without loss of generality assume that $u_{(n+1)j} = u_{1j}$. Then we have

$$\sum_{i=2}^n \sum_{j=2}^m \ell^+(u_{ij}) + \sum_{j=1}^m \ell^+(u_{1j}) = 0$$

This implies that $[(m-1)n+1]$ is even which again implies that m is even and n is odd. Conversely assume that m is even and n is odd. Define a labeling $\ell : E(T_n^{(m)}) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} & \text{For } i = 1, 3, \dots, n \text{ do :} \\ & \ell(u_{ij}u_{(j+1)}) = \begin{cases} b, & j = 1, 3, \dots, m-1 \\ c, & j = 2, 4, \dots, n \end{cases} \\ & \text{end for} \\ & \text{For } i = 2, 4, \dots, n-1 \text{ do :} \\ & \ell(u_{ij}u_{(j+1)}) = \begin{cases} c, & j = 2, 3, \dots, m-2 \\ b, & j = 1, m-1, m \end{cases} \\ & \text{end for} \end{aligned}$$

Clearly $\ell^+(v) = a$ for all $v \in V(T_n^{(m)})$. $\square$

Theorem 2.28. [Sweetly & Paulraj Joseph (2010)] $T_n^{(m)} \in \mathcal{V}_0$ for all n and m .

Theorem 2.29. $T_n^{(m)} \in \mathcal{V}_{a,0}$ if and only if m is even and n is odd.

Proof. The proof follows from theorems 2.27 and 2.28. □

Definition 2.9 (Sweetly & Paulraj Joseph (2010)). The cartesian product of graphs P_m and P_n denoted, $P_m \square P_n$ is called a planar grid.

The planar grid $P_6 \square P_4$ is shown in Figure 5.

Theorem 2.30. The planar grid $P_m \square P_n$ is a -sum V_4 magic if and only if mn is even.

Proof. Suppose that $P_m \square P_n \in \mathcal{V}_a$. Let $(i, j), i = 0, 1, \dots, m-1, j = 0, 1, \dots, n-1$ denote the vertices of $P_m \square P_n$. We have $\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \ell^+((i, j)) = 0$. Thus we have mn is even. For the converse consider the following cases.

Case 1: Both m and n are even.

Define a labeling $\ell : E(P_m \square P_n) \rightarrow V_4 \setminus \{0\}$ as follows.

```

For  $i = 0, 1, \dots, m-2$  do :
 $\ell((i, j)(i+1, j)) = b, j = 0, n-1$ 
end for
For  $j = 0, 1, \dots, n-2$  do :
 $\ell((i, j)(i, j+1)) = c, i = 0, m-1$ 
end for
For  $i = 1, 2, \dots, m-2$  do :
 $\ell((i, j)(i, j+1)) = \begin{cases} a, & j = 0, 2, \dots, n-2 \\ c, & j = 1, 3, \dots, n-3 \end{cases}$ 
end for
For  $j = 1, 2, \dots, n-2$  do :
 $\ell((i, j)(i+1, j)) = \begin{cases} a, & i = 0, 2, \dots, m-2 \\ b, & i = 1, 3, \dots, m-3 \end{cases}$ 
end for

```

With this labeling $P_m \square P_n$ is a -sum V_4 magic.

Case 2: m is even and n is odd.

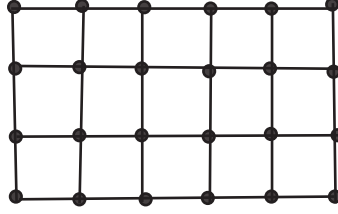


Figure 5: Planar grid $P_6 \square P_4$

Define a labeling $\ell : E(P_m \square P_n) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$\begin{aligned}
 \ell((i, 0)(i + 1, 0)) &= b, \quad i = 0, 1, \dots, m - 2 \\
 \ell((i, n - 1)(i + 1, n - 1)) &= \begin{cases} b, & i = 0, 2, \dots, m - 2 \\ a, & i = 1, 3, \dots, m - 3 \end{cases} \\
 &\text{For } j = 0, 1, \dots, n - 2 \text{ do :} \\
 \ell((i, j)(i, j + 1)) &= c, \quad i = 0, m - 1 \\
 &\text{end for} \\
 &\text{For } j = 1, 2, \dots, n - 2 \text{ do :} \\
 \ell((i, j)(i + 1, j)) &= \begin{cases} a, & i = 0, 2, \dots, m - 2 \\ c, & i = 1, 3, \dots, m - 3 \end{cases} \\
 &\text{end for} \\
 &\text{For } i = 1, 2, \dots, m - 2 \text{ do :} \\
 \ell((i, j)(i, j + 1)) &= \begin{cases} a, & j = 0, 2, \dots, n - 3 \\ b, & j = 1, 3, \dots, n - 2 \end{cases} \\
 &\text{end for}
 \end{aligned}$$

Obviously $P_m \square P_n$ is a -sum V_4 magic.

Case 3: m is odd and n is even.

By interchanging the roles of m and n in Case 2, we get $\ell^+(v) = a$ for all $v \in V(P_m \square P_n)$.

This completes the proof. □

Theorem 2.31. [Sweetly & Paulraj Joseph (2010)] $P_m \square P_n \in \mathcal{V}_0$ for all m and n .

Theorem 2.32. $P_m \square P_n \in \mathcal{V}_{a,0}$ if and only if mn is even.

Proof. The proof follows from theorems 2.30 and 2.31. □

Theorem 2.33. For $m, n \geq 2$, the complete bipartite graph $K(m, n)$ is a -sum V_4 magic if and only if $m + n$ is even.

Proof. First assume that $K(m, n)$ is a -sum V_4 magic. Let $\{u_i, i = 1, 2, \dots, m\} \cup \{v_j, j = 1, 2, \dots, n\}$ be the vertices of the graph with $E(G) = \{u_i v_j : i = 1, 2, \dots, m, j = 1, 2, \dots, n\}$. Then we have $\sum_{i=1}^m \ell^+(u_i) + \sum_{j=1}^n \ell^+(v_j) = 0$. This implies that $m + n$ is even. Conversely suppose that $m + n$ is even. Then we have the following cases.

Case 1: m and n are odd.

Define a labeling $\ell : E(K(m, n)) \rightarrow V_4 \setminus \{0\}$ by

For $i = 1, 2, \dots, m$ do :
 $\ell(u_i v_j) = a, j = 1, 2, \dots, n.$
 end for

Thus $\ell^+(v) = a$.

Case 2: m and n are even.

$\ell(u_i v_2) = c, i = 1, 3, 4, \dots, m,$
 $\ell(u_2 v_j) = c, j = 1, 3, 4, \dots, n,$
 $\ell(u_2 v_2) = b,$
 For $i = 1, 3, 4, \dots, m$ do :
 $\ell(u_i v_j) = b, j = 1, 3, 4, \dots, n.$
 end for

Obviously, $K(m, n)$ is a -sum V_4 magic. □

Theorem 2.34. [Lee et al. (2002)] $K(m, n)$ is zero-sum V_4 magic for all m and n .

Theorem 2.35. $K(m, n) \in \mathcal{V}_{a,0}$ if and only if $m + n$ is even.

Proof. The proof follows from theorems 2.33 and 2.34. □

Definition 2.10 (Lee et al. (2002)). When k copies of C_n share a common edge it will form the n -gon book of k pages and is denoted by $B(n, k)$.

The graph $B(6, 4)$ is shown in Figure 6.

Lemma 2.36. If ℓ is an a -sum V_4 magic labeling of $B(n, k)$, then

$$\sum_{i=1}^{n-2} \sum_{j=1}^k \ell^+(u_i^j) + \ell^+(u) + \ell^+(v) = 0$$

where u and v are the common vertices and $u_1^j, u_2^j, \dots, u_{n-2}^j$ are the other vertices of the j^{th} page.

Proof. The proof is similar to Lemma 2.1. □

Theorem 2.37. For any $n \geq 3$ and $k \geq 1$, $B(n, k) \in V_a$ if and only if $(n - 2)k$ is even.

Proof. First assume that $B(n, k) \in V_a$. Then $[(n - 2)k + 2]a = 0$. This implies that $(n - 2)k$ is even. Conversely assume that $(n - 2)k$ is even. We consider the following cases.

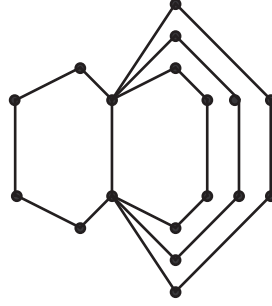


Figure 6: The graph $B(6, 4)$

Case 1: Both n and k are even.

Define a labeling $\ell : E(B(n, k)) \rightarrow V_4 \setminus \{0\}$ by

$$\ell(uv) = a,$$

For $j = 1, 2, \dots, k$ do :

$$\ell(uu_1^j) = \ell(vu_{(n-2)}^j) = b,$$

$$\ell(u_{2i}^j u_{2i+1}^j) = b, \text{ for } i = 1, 2, \dots, \frac{n-4}{2},$$

$$\ell(u_{2i-1}^j u_{2i}^j) = c, \text{ for } i = 1, 2, \dots, \frac{n-4}{2}.$$

end for

Then ℓ is an a -sum V_4 magic labeling of $B(n, k)$.

Case: 2 n is odd and k is even,

Define a labeling $\ell : E(B(n, k)) \rightarrow V_4 \setminus \{0\}$ by

$$\ell(uv) = a,$$

For $j = 1, 2, \dots, k$ do :

$$\ell(uu_1^j) = b,$$

$$\ell(vu_{(n-2)}^j) = c,$$

$$\ell(u_{2i}^j u_{2i+1}^j) = b, \text{ for } i = 1, 2, \dots, \frac{n-3}{2},$$

$$\ell(u_{2i-1}^j u_{2i}^j) = c, \text{ for } i = 1, 2, \dots, \frac{n-3}{2}.$$

end for

Case 3: n is even and k is odd

Define a labeling $\ell : E(B(n, k)) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(uv) &= c, \\ \text{For } j = 1, 2, \dots, k \text{ do :} \\ \ell(uw_1^j) &= \ell(vw_{(n-2)}^j) = b, \\ \ell(u_{2i}^j u_{2i+1}^j) &= b, \text{ for } i = 1, 2, \dots, \frac{n-4}{2}, \\ \ell(u_{2i-1}^j u_{2i}^j) &= c, \text{ for } i = 1, 2, \dots, \frac{n-2}{2}. \\ \text{end for} \end{aligned}$$

Thus ℓ is an a -sum V_4 magic labeling of $B(n, k)$. $\square$

Theorem 2.38. [Lee et al. (2002)] For any $n \geq 3$ and $k \geq 1$, $B(n, k)$ is zero-sum V_4 magic.

Theorem 2.39. For any $n \geq 3$ and $k \geq 1$, $B(n, k) \in \mathcal{V}_{a,0}$ if and only if $(n-2)k$ is even.

Proof. The proof follows from theorems 2.37 and 2.38. $\square$

Definition 2.11 (Sweetly & Paulraj Joseph (2010)). The book B_n is the graph $S_n \square P_2$ where S_n is the star with $n+1$ vertices.

The book graph B_4 is shown in Figure 7.

Theorem 2.40. The book B_n is a -sum V_4 magic for all n .

Proof. Let w_1, w_2 be the vertices of the common edge. Let $\{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_n\}$ be the vertices of B_n .

Case 1: n is odd.

$$\begin{aligned} \ell(w_1 w_2) &= c, \\ \text{For } i = 1, 2, \dots, n \text{ do :} \\ \ell(w_1 u_i) &= \ell(w_2 v_i) = b, \\ \ell(u_i v_i) &= c. \\ \text{end for} \end{aligned}$$

Case 2: n is even.

$$\begin{aligned} \ell(w_1 w_2) &= a, \\ \text{For } i = 1, 2, \dots, n \text{ do :} \\ \ell(w_1 u_i) &= \ell(w_2 v_i) = b, \\ \ell(u_i v_i) &= c. \\ \text{end for} \end{aligned}$$

Clearly ℓ is an a -sum V_4 magic labeling of B_n . $\square$

Theorem 2.41. $B_n \in \mathcal{V}_0$ for all n .

Proof. We consider the following cases:

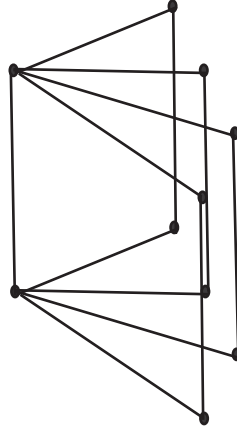


Figure 7: Book graph B_4

Case 1: n is odd.

Label all the edges by a . Then we get $\ell^+(v) = 0$ for all $v \in V(G)$.

Case 2: n is even.

Define a labeling $\ell : E(B_n) \rightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}\ell(w_1w_2) &= a, \\ \ell(u_1w_1) &= \ell(v_1w_1) = \ell(u_1v_1) = c, \\ \ell(w_1u_i) &= \ell(w_2v_i) = b, i = 2, 3, \dots, n, \\ \ell(u_iv_i) &= b, i = 2, 3, \dots, n.\end{aligned}$$

With this labeling we get $\ell^+(v) = 0$ for all $v \in V(G)$.

□

Definition 2.12 (Lee et al. (2002)). Given a cycle C_n construct a cycle C_m on each edge of this cycle. The resulting graph is called flower graph and is denoted by $C_m @ C_n$.

Theorem 2.42. For all $m, n \geq 3$, the flower graph $C_m @ C_n \in \mathcal{V}_a$ if and only if $n(m-1)$ is even.

Proof. Suppose that $C_m @ C_n \in \mathcal{V}_a$. Then $[n(m-1)]a = 0$. This implies that $n(m-1)$ is even. Now let $u_1, u_2, \dots, u_n$ be the vertices of C_n and $v_{1j}, v_{2j}, \dots, v_{(m-2)j}$ be the vertices of $C_j, j = 1, 2, \dots, n$. We consider the following cases.

Case 1: n is even and m is odd.

For $j = 1, 2, \dots, n$ do :

$$\begin{aligned} \ell(u_j u_{(j+1)}) &= a, \\ \ell(u_j v_{1j}) &= c, \\ \ell(u_{(j+1)} v_{(m-2)j}) &= b, \\ \ell(v_{ij} v_{i(j+1)}) &= b, \quad i = 1, 3, \dots, m-4, \\ \ell(v_{ij} v_{i(j+1)}) &= c, \quad i = 2, 4, \dots, m-3. \end{aligned}$$

end for

Case 2: Both n and m are even.

$$\begin{aligned} \ell(u_j u_{(j+1)}) &= a, \quad j = 1, 2, \dots, n, \\ \text{For } j &= 1, 3, \dots, n-1 \text{ do :} \\ \ell(v_{ij} v_{i(j+1)}) &= c, \quad i = 1, 3, \dots, m-3, \\ \ell(v_{ij} v_{i(j+1)}) &= b, \quad i = 2, 4, \dots, m-4. \\ \text{end for} \\ \text{For } j &= 2, 4, \dots, n \text{ do :} \\ \ell(v_{ij} v_{i(j+1)}) &= b, \quad i = 1, 3, \dots, m-3, \\ \ell(v_{ij} v_{i(j+1)}) &= c, \quad i = 2, 4, \dots, m-4. \\ \text{end for} \end{aligned}$$

Case 3: Both n and m are odd.

For $j = 1, 2, \dots, n$ do :

$$\begin{aligned} \ell(u_j u_{(j+1)}) &= a, \\ \ell(u_j v_{1j}) &= b, \\ \ell(u_{(j+1)} v_{(m-2)j}) &= c, \\ \ell(v_{ij} v_{i(j+1)}) &= c, \quad i = 1, 3, \dots, m-4, \\ \ell(v_{ij} v_{i(j+1)}) &= b, \quad i = 2, 4, \dots, m-3. \end{aligned}$$

end for

Thus ℓ is an a -sum V_4 magic labeling of $C_m @ C_n$. This completes the proof. $\square$

Theorem 2.43. [Lee et al. (2002)] For all $m, n \geq 3$, the flower graph $C_m @ C_n \in \mathcal{V}_0$.

Theorem 2.44. For all $m, n \geq 3$, the flower graph $C_m @ C_n \in \mathcal{V}_{a,0}$ if and only if $n(m-1)$ is even.

Proof. The proof follows from theorems 2.42 and 2.43. $\square$

Definition 2.13 (Lee et al. (2002)). Given k natural numbers $a_1, a_2, \dots, a_k$, if we connect the two vertices of $N_2 = \{u, v\}$ by k parallel paths of length $a_1, a_2, \dots, a_k$, the resulting graph is called the generalized Theta graph and is denoted by $\Theta(a_1, a_2, \dots, a_k)$.

Note that in this graph, $\deg(u) = \deg(v) = k$ and all the other vertices are of degree two.

Theorem 2.45. If the generalized Theta graph $\Theta(a_1, a_2, \dots, a_k)$ is a -sum V_4 magic then either odd number of a_i 's are odd or even number of a_i 's are even.

Proof. First suppose that $\Theta(a_1, a_2, \dots, a_k)$ is a -sum V_4 magic. Then we have $\left[\sum_{i=1}^k a_i - k \right] a = 0$. This implies that $\left[\sum_{i=1}^k a_i \right] a = ka$. This is if and only if both $\sum_{i=1}^k a_i$ and k are odd or even simultaneously. This happens if and only if odd number of a_i 's are odd or even number of a_i 's are even. $\square$

Theorem 2.46. *Let $\Theta(a_1, a_2, \dots, a_k)$ be a generalized Theta graph. If k and even number of a_i 's are even then $\Theta(a_1, a_2, \dots, a_k)$ is a -sum V_4 magic.*

Proof. Let $v_1^i, v_2^i, \dots, v_{a_i-1}^i$ be the vertices of the i^{th} path and let u, w be the common vertices. Define a labeling $\ell : \Theta(a_1, a_2, \dots, a_k) \rightarrow V_4 \setminus \{0\}$ by

For $i = 1, 2, \dots, k-1$ do :

$$\ell(uv_1^i) = b$$

end for

$$\ell(uv_1^k) = c$$

For $i = 1, 2, \dots, k$ do :

$$\ell(v_j^i v_{j+1}^i) = \begin{cases} c, & j = 1, 3, \dots, a_i - 2, \text{ if } a_i \text{ is odd} \\ j = 1, 3, \dots, a_i - 3, \text{ if } a_i \text{ is even} \end{cases}$$

$$\ell(v_j^i v_{j+1}^i) = \begin{cases} b, & j = 2, 4, \dots, a_i - 2, \text{ if } a_i \text{ is even} \\ j = 2, 4, \dots, a_i - 3, \text{ if } a_i \text{ is odd} \end{cases}$$

Now label the edge $v_{a_k-1}^k w, i = 1, 2, \dots, k$ in the following way.

If $\ell(v_{a_k-2}^k v_{a_k-1}^k) = b$, let $\ell(v_{a_k-1}^k w) = c$ and viceversa. Thus ℓ is an a -sum V_4 magic labeling of $\Theta(a_1, a_2, \dots, a_k)$. $\square$

Theorem 2.47. [Lee et al. (2002)] $\Theta(a_1, a_2, \dots, a_k)$ is zero-sum V_4 magic for any sequence $a_1, a_2, \dots, a_k$.

Theorem 2.48. *Let $\Theta(a_1, a_2, \dots, a_k)$ be a generalized Theta graph. If k and even number of a_i 's are even then $\Theta(a_1, a_2, \dots, a_k) \in \mathcal{V}_{a,0}$.*

Proof. The proof follows from theorems 2.46 and 2.47. $\square$

Definition 2.14 (Lee et al. (2002)). Let $N_2 = \{v_1, v_2\}$ be the disconnected graph of order two. The graph $C_n \vee N_2$ is called the bipyramid based on C_n and is denoted by $BP(n)$.

The bipyramid graph based on C_6 denoted as $BP(6)$ is shown in Figure 8.

Theorem 2.49. *For any $n \geq 4$, the bipyramid graph $BP(n)$ is a -sum V_4 magic if and only if n is even.*

Proof. Suppose that $BP(n) \in \mathcal{V}_a$. This implies that $(n+2)a = 0$. Thus we have obtained n is even. Conversely assume that n is even.

For $i = 1, 2$ do :

$$\ell(v_i u_1) = c,$$

$$\ell(v_i u_j) = b, j = 2, 3, \dots, n$$

end for

$$\ell(u_j u_{(j+1)}) = b, j = 1, 3, \dots, n-1,$$

$$\ell(u_j u_{(j+1)}) = c, j = 2, 4, \dots, n.$$

Obviously $BP(n) \in \mathcal{V}_a$. $\square$

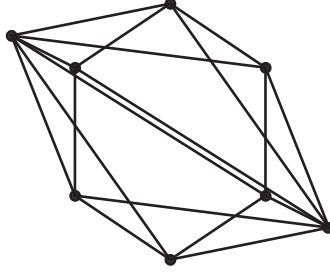


Figure 8: The graph $BP(6)$

Theorem 2.50. [Lee et al. (2002)] For any $n \geq 4$, $BP(n)$ is zero-sum V_4 magic for all n .

Theorem 2.51. For any $n \geq 4$, $BP(n) \in \mathcal{V}_{a,0}$ if and only if n is even.

Proof. The proof follows from theorems 2.49 and 2.50. □

Definition 2.15 (Lee et al. (2002)). Given a graph G , we can define the bypyramid based on G to be $G \vee N_2$. This graph will be denoted by $BP(G)$.

Theorem 2.52. Consider the bypyramid graph $BP(G)$ based on G . We have the following.

- i) If G is α -sum V_4 magic and number of vertices in G is odd, then $BP(G)$ is α -sum V_4 magic.
- ii) If G is α -sum V_4 magic and number of vertices in G is even, then $BP(G)$ is 0-sum V_4 magic.
- iii) If G is 0-sum V_4 magic and number of vertices in G is even, then $BP(G)$ is both α -sum V_4 magic and 0-sum V_4 magic.
- iv) If G is 0-sum V_4 magic and number of vertices in G is odd, then $BP(G)$ is 0-sum V_4 magic.

Proof. Obvious. □

Corollary 2.53. We have the following.

- i) If G is 0-sum V_4 magic, then $G \vee \overline{K_n} \in \mathcal{V}_{a,0}$.
- ii) If G is α -sum V_4 magic, then $G \vee \overline{K_n} \in \mathcal{V}_{a,0}$.

3 CONCLUSIONS

This paper is a continuation of the work carried out in [Anil Kumar & Vandana (2015)] and [Vandana & Anil Kumar (2015)]. In this paper we investigated graphs in the following categories:

- 1) $\mathcal{V}_\alpha$, the class of α -sum V_4 magic graphs.
- 2) $\mathcal{V}_0$, the class of zero-sum V_4 magic graphs.

3) $\mathcal{V}_{a,0}$, the class of graphs which are both a -sum and zero-sum V_4 magic.

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Competing Interests

The authors declare that no competing interests exist.

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