A New Approach to Output Control of Smart Beams under Axial and Transverse Loads

Mark A Pinsky[*1] **and Steve Koblik**[1]

[1] *Department of Mathematics and Statistics,*
University of Nevada.Reno,
Reno Nevada, USA

Abstract

A problem of suppressing vibrations of elastic structures through controlling their finite-dimensional approximations was addressed in various publications which primarily utilize collocated sensors and actuators. This engineering approach, however, raised concerns that a control designed for a finite dimensional model could excite or destabilize its abandoned infinite-dimensional complement, which reflects in so known spillover phenomenon. This paper developed a novel design of an output controller, which eliminates spillover in control and marginalizes its contribution to observer design. This consequently leads to an efficient control of beam vibrations, which can be attained by non-collocated sensors and actuators located nearly anywhere on the beam. We show in simulations that the proposed methodology leads to stabilization of post-critical buckling and an efficient reduction of the beam's resonance vibrations enforced by loads comprised of an axial force and unknown transverse bounded disturbances. We infer controllability and observability conditions which pinpoint areas on the beam where locating actuators/sensors should be avoided and estimated the contribution of discarded part of the initial system.

Keywords: Output control, smart beams, control and measurement spillovers, vibration control, buckling control

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1 Introduction

A problem of vibration control of smart piezoelectric beams and other smart structures has been studied extensively in the past decades; see, for example, review papers [1-3] in the engineering literature as well as references [4-13] in the distributed parameter systems literature. A more general problem in controlling PDEs [14] and the dynamics of elastic structures has been addressed in a number of recent publications, for example, in [15-16]. Vibration control of beams and columns loaded

**Corresponding author: E-mail: pinsky@unr.edu*

by combination of axial and transverse forces have been studied as well; see [6] for recent review of relevant publications. Design of output vibration controllers for flexible structures were developed in a number of studies[17-19], where additional references can be found. Different approaches to the design of observers for linear PDE-systems were surveyed in [20]. Lesser attention has been given to the estimation of the impact of spillover on the design of output controllers.

There are two major thrusts for the design of estimators and controllers for PDEs. One designs the control for a finite-dimensional ODE-system, approximating the PDE-model, while the other embraces control for the entire infinite dimensional system. The former approach is used frequently, since it enables applications of design techniques readily available for LTI-systems. At the same time, this approach is subject to observation and control spillovers [21], which may result in destabilizing or exciting a discarded counterpart of the initial infinite-dimensional system and degrade performance of the observer and controller. That and subsequent studies [22-23] revealed some conditions under which stability of reduced system implies stability of initial PDEs; however, treacherous dynamics triggered by control forces in the voided infinite dimensional part of the initial system has not been rigorously estimated. Another mentioned drawback of this approach is that it sometimes renders unnatural feedback laws which are difficult to interpret [24].

It has been shown that applications of high-gain observers to linear and some nonlinear systems of ODEs lead to the reduction of the estimation error caused by bounded uncertainties and enhances performance of output controllers[25-26]; however, accounting for measurement noise limits the efficacy of these observers. Different approaches have been proposed to mitigate the effect of noise on the observer performance. These approaches were primarily designed for systems of ODEs, see [27-30] and additional references therein.

Disturbance decoupled observers utilizing a linear feedback control have been developed for a special class of second order PDEs in [31], where additional references on this subject can be also found. This design, however, requires knowledge of the operator associated with spatial distribution of the disturbances.

This paper develops a novel approach to output control of PDEs that is based on finite dimensional approximation of these models and applies it to the vibration control of a simply supported smart beam/column which is equipped with piezo-sensors and actuators and loaded with axial and dynamic transverse forces. The proposed design voids controller spillover and marginalizes the effect of observer spillover on the performance of closed-loop systems. The efficiency of this approach, in stabilizing and reducing forced smart-beams vibrations, is demonstrated in representative simulations. We show that control of this beam's vibrations can be attained by near arbitrary locations of its sensors and actuators and scale dynamics of uncontrolled infinite dimensional counterpart of the initial system.

2 Smart Beam Model

The Si system has been adopted throughout this paper for all variables and parameters. We consider an Euler-Bernoulli beam of length L, height $h = const$ and width $b = const$ equipped with a number of piezoelectric patches with equal height $\hbar$ which are firmly bonded onto the beam's top face. Such patch can be used as an actuator or a sensor. Let the lengths of actuator and sensor-patches be a_i and s_i and their left edges are located at x_i^a and x_i^s, $i = 1, ..., m$ respectively. It is commonly assumed that $\hbar \ll h$ and that contribution of mechanical properties of patches and bounding material on beam dynamics is negligible [6]. Let the beam is located on interval $[0, L]$ and its axial and transverse loads are $T = const$ and $F = F(x, t)$, and actuator forces f_i are uniformly distributed along i-th patch and equal zero elsewhere.

We write the equation of motion of such beam as follows:

$$EI\frac{\partial^4 w}{\partial x^4}+\beta\frac{\partial^5 w}{\partial x^4\partial t}+T\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial t^2}-\sum_{i=1}^{m}f_i=F(x,t) \tag{2.1}$$

where $w(x,t), E, I\beta, \rho$ are the beam's transverse displacement, Young's modulus, momentum of inertia, damping coefficient and mass density; $\mu=\rho bh$ is the mass per unit length. To determine f_i , let us first write the strain of piezoelectric patch as $\epsilon_i=-V_i d/\hbar$, where d and $V_i(t)$ are the piezoelectric constant and voltage on the i-th patch; then $\sigma_i=E_p\epsilon_i$ and E_p are the stress and Young's modulus of an i-th-patch, and $M_i=\sigma_i b\hbar(h+\hbar)\chi_i$ its momentum, where the characteristic function of the patch, $\chi_i=H(x_i^a)-H(x_i^a+a_i)$, and H is a Heaviside function. Finally, $f_i=\partial^2 M_i/\partial x^2$. Note that more details on this derivation can be found in [6]. We assume that $F(x,t)$ is twice continuously differential in x but can have finite number of jumps with finite magnitude on any finite time interval.

The hinged boundary conditions for this equation are

$$w(0,\,t)=w(L,t)=\partial^2 w/\partial x^2(0,\,t)=\partial^2 w(L,t)/\partial x^2=0 \tag{2.2}$$

We assume that the transversal force also satisfies the hinged boundary conditions, and that the initial displacements and velocities of the beam are twice continuously differentiable functions:

$$w(x,0)=w^0(x),\quad \dot{w}(x,0)=w^1(x),\, x\in[0,L] \tag{2.3}$$

Note that [4] proves well-posedness of these type of equations which are normally considered in the weak form, see [32] for further details on this subject.

3 Modal Equations

We shall associate to the problem (2.1)-(2.3), the Sturm-Liouville problem

$$W''''=\sigma^4 W, x\in(0,L)$$
$$W(0)=W''(0)=W(L)=W''(L)=0$$

whose eigenvalues and normalized eigenfunctions are: $\sigma_n=n\pi/L$ and $\psi_n(x)=\sqrt{2}sin\sigma_n x, n=1,2...$

Now setting

$$w(x,t)=\sum_{j=1}^{\infty}w_j(t)\psi_j(x),\quad F(x,t)=\sum_{j=1}^{\infty}F_j(t)\psi_j(x)$$

substituting the last expression into (2.1), using orthogonality condition and integrating twice by parts the products of actuator and eigenfunctions terms, we get the following system of ODEs :

$$\mu\ddot{w}_n+\sigma_n^2(EI\sigma_n^2-T)w_n+\sigma_n^4\beta\dot{w}_n=\gamma\sum_{i=1}^{m}V_i(t)(\psi_n^{'}(x_i^a)-\psi_n^{'}(x_i^a+a_i))+F_n(t), n=1,2,...$$

where $\gamma=E_p db(h+\hbar)$, and $\dot{w}_n$, and $\psi_n^{'}$ are temporal and spatial derivatives of respected functions.

3.1 Controller Design

We introduce a feedback controller in the following form

$$V_i(t)=\int_0^L k_i^1(x)w(x,t)\,dx+\int k_i^2(x)\dot{w}(x,t)\,i=1,...,m$$

where continuously differential gain functions $k_i^1(x)$ and $k_i^2(x)$, obey the hinged boundary conditions as well. Thus expending $k_i^{1,2}(x)$ in convergent series of eigenfunctions:$\{\psi_n\}_{n=1}^{\infty}$; one gets:

$$V_i(t) = \int_0^L \left(\sum_{j=1}^{\infty} k_{ij}^1 \psi_j(x)\right)\left(\sum_{j=1}^{\infty} w_j(t)\psi_j(x)\right)\, dx + \int_0^L \left(\sum_{j=1}^{\infty} k_{ij}^2 \psi_j(x)\right)\left(\sum_{j=1}^{\infty} \dot{w}_j(t)\psi_j(x)\right) dx$$

Evoking orthogonality conditions, we write

$$V_i(t) = \sum_{j=1}^{\infty}(k_{ij}^1 w_j + k_{ij}^2\, \dot{w}_j)$$

where $k_{ij}^{1,2}$ are adjustable control gains. Now pretend that we are going to control N_1 beam's modes and leave the rest uncontrolled. Then setting $k_{ij}^{1,2} = 0$ for $j > N_1$, we write the previous formula as

$$V_i(t) = \sum_{j=1}^{N_1}(k_{ij}^1 w_j(t) + k_{ij}^2 \dot{w}_j),\ i \dot{=} 1, ..., m$$

Note that the last formula can express control force as a linear combination of any chosen number of the beam's eigenfunctions or modes, while eliminating projections of control forces on the rest of the modes. This naturally voids spillover in control. However, as a rule, the beam's modes are not directly measurable which necessitates observer design.

3.2 Sensor Equations

We assume that m-piezoelectric sensors are distributed over the beam and the voltage on each sensor $y_i(t)$ is measurable. To educe voltage dependence upon $w(x,t)$ we adopt from [6], the customary laws which in this case can be written as follows:

$$y_i(t) = \nu(w^{'}(x_i^s) - w^{'}(x_i^s + s_i)) + \varepsilon_i(t), i = 1, ..., m$$

where $\nu = \hbar e_{31}(h+\hbar)/2s_i e_{33}$, e_{31} and e_{33} are the permittivity and piezoelectric stress constants, see [6] for more details, and $\varepsilon_i(t)$ accounts for bounded measurement noise,$\|\varepsilon_i\| < \varepsilon_0$. At the edges of an actuator's patch,f_i , and hence $\partial w^4/\partial x^4$, are represented as x-derivatives of Dirac's delta function, i.e, $f_i = \sigma_i b\hbar(h+\hbar)d^2\chi_i/dx^2$. Thus $w^{'}(x,t)$ is continuous on the interval $[0, L]$ and expending $w^{'}(x,t)$ in series of eigenfunctions, we write the last formula as:

$$y_i(t) = \nu \sum_{j=1}^{N_1} w_j(t)(\psi_j^{'}(x_i^s) - \psi_j^{'}(x_i^s + s_i)) + \mathcal{O}(1/(N_1+1) + \varepsilon_i$$

Thus the residual of their partial sums, accounting for observer's spillover, gets arbitrarily small if $N_1 \to \infty$, and will be treated as a component of bounded measurement noise in the reminder of this paper.

3.3 State Equations

Let's write controlled and uncontrolled state vectors as: $z_1 = [\dot{w}_1, ..., \dot{w}_{N_1},\ w_1, ..., w_{N_1}]$ and $z_1 = [\dot{w}_{N_1+1}, ..., \dot{w}_N,\ w_{N_1+1}, ..., w_N]$. Then the state equations can be written in the form:

$$\dot{z}_1 = Q_1 z_1 + BV + \mathcal{F}_1(t)$$

$$\dot{z}_2 = Q_2 z_2 + \mathcal{F}_2(t)$$

$$y = Cz_1 + \varepsilon$$

where

$$Q_{1=}\begin{bmatrix} O_1 & I \\ -A_1 & -\alpha\Theta_1 \end{bmatrix}, Q_2 = \begin{bmatrix} O_2 & I \\ -A_2 & -\alpha\Theta_2 \end{bmatrix}$$

$$A_1 = (1/\mu)\mathrm{diag}((\sigma_1^2)(EI\sigma_1^2 - T), \cdots, (\sigma_{N_1}^2)(EI\sigma_{N_1}^2 - T))$$

$$A_2 = (1/\mu)diag(\sigma_{N_1+1}^2)(EI\sigma_{N_1+1}^2 - T), ..., \sigma_N^2(EI\sigma_N^2 - T)$$

$$\Theta_1 = diag(\sigma_1^4, ..., \sigma_{N_1}^4), \Theta_{2=}\mathrm{diag}(\sigma_{N_1+1}^4, \cdots, \sigma_N^4)$$

$O_i,\ i = 1, 2$ are zero-matrices of appropriate dimensions, I is an identity matrix, $\alpha = \beta/\mu$, $2N_1 \times m-$matrix $B = \{b_{ij}\}$, $b_{ij} = \psi_i^{'}(x_j^a) - \psi_i^{'}(x_j^a + a_j)$, $i = N_1 + 1, ..., 2N_1$, $j = 1, ..., m$, and $b_{ij} = 0$ otherwise; $\mathcal{F}_1 = [0, ..., 0, F_1, ..., F_{N_1}]^T$, $\mathcal{F}_2 = [0, ..., 0, F_{N_1+1}, ..., F_N]^T$, and $C = \{c_{ij}\}$, $c_{ij} = \nu(\psi_j^{'}(x_i^s) - \psi_j^{'}(x_i^s + s_i))$, $i = 1, ..., m$, $j = N_1 + 1, ..., N$, $c_{ij} = 0$ for $\forall j : 1 \leq j \leq N_1$,

$y = \{y_i\}, \varepsilon = \{\varepsilon_i\}, i = 1, ..., m.$

3.4 Output Controller Equations

Assuming that vectors $\mathcal{F}_i,\ i = 1, 2$ are unknown but norm-bounded, we write the observer equation as

$$\dot{\xi} = Q_1\xi + BV + L(y - C\xi) + L\varepsilon$$

Then the error equation becomes:

$$\dot{e} = (Q_1 - LC)e - L\varepsilon + \mathcal{F}_1(t) \tag{3.1}$$

where $e = z_1 - \xi$ and L is observer gain matrix.

Now set $V = -K\xi$, where K is a $m \times 2N_1$ control gain matrix. Then writing $V = -K(z_1 - e)$, we get:

$$\dot{z}_1 = (Q_1 - BK)z_1 + BKe + \mathcal{F}_1(t) \tag{3.2}$$

Note that the error equation (3.1) is decoupled from state equation (3.2), and these two equations yield the output controller.

4 Controllability and Observability Conditions

4.1 Controllability Conditions

We are going to derive controllability condition for our system using modal controllability criterion requiring diagonalization of system matrix Q_1 which we write now in the block-diagonal form:

$$Q_1 = diag(\Upsilon_1, ..., \Upsilon_{N_1})$$

, where $\Upsilon_i = (1/\mu)\begin{bmatrix} 0 & 1 \\ -\sigma_i^2(EI\sigma_i^2 - T) & -\alpha\sigma_i^4 \end{bmatrix}$. Let the eigenvalues and eigenvectors of the last matrix be: $\lambda_{i1,2}$ and $[1\,\lambda_{i1,2}]^T$ respectively. Then the matrix diagonalizing Υ_i can be written as $d_i = \begin{bmatrix} 1 & 1 \\ \lambda_{i1} & \lambda_{i2} \end{bmatrix}$. In turn, matrix B can be also written as $B = diag(B_1, ..., B_{N_1})$, where $B_i = \begin{bmatrix} 0 & ... & 0 \\ b_{i1} & ... & b_{im} \end{bmatrix}$, $b_{ij} = \psi_i^{'}(x_j^a) - \psi_i^{'}(x_j^a + a_j)$. Thus $d_i^{-1}B_i = \Delta_i^{-1}\begin{bmatrix} -b_{i1} & ... & -b_{im} \\ b_{i1} & ... & b_{im} \end{bmatrix}$, Δ_i

is the determinant of matrix d_i , and $i = 1, ...N_1$, $j = 1, ..., m$. Due to modal criterion the pair (Q_1, B) is controllable if all matrices $d_i^{-1}B_i$ have no zero rows, i.e. $\forall i = 1, ..., N_1 | \exists\, j = 1, .., m$ such that $\psi_i^{'}(x_j^a) \neq \psi_i^{'}(x_j^a + a_j)$. This implies that $\forall i = 1, ..., N_1 | \exists\, j = 1, .., m$ such that $cos(i\pi x_j^a) \neq cos(i\pi x_j^a + a_j)$ which finally yields the following controllability conditions.

Theorem 4.1. *Pair (Q_1, B) is controllable if for every $i = 1, ..., N_1$ and $j = 1, ..., m$ and positive integers $k_1 \leq i/2$ and $k_2 \leq i$, there are at least one $a_j \neq 2Lk_1/i$ and at least one $x_j^a + a_j/2 \neq Lk_2/j$, i.e.:*

$$\forall i = 1, ..., N_1 \,\&\, j = 1, ..., m \,\&\, k_1 \& \, k_2 \in \mathbb{N} \,|\, \&\, k_1 \leq i/2 \, k_2 \leq i | \exists\, a_j \neq 2Lk_1/i \,\&\, x_j^a + a_j/2 \neq Lk_2/j \tag{4.1}$$

It is clear that the restrictions imposed by these conditions on lengths and locations of piezoelectric actuators are lessening if their number raises. It follows from (4.1) that if there is only one actuator, it still can be placed almost everywhere on the beam except indicated locations and might have almost arbitrary length which, however, should comply with restrictions imposed by Theorem 1.

4.2 Observability Conditions

We derive observability conditions by application of modal criterion as well. As previously, we will keep matrix Q_1 and its diagonalizing matrix in block diagonal forms. For this setting matrix $C = \{c_{ij}\}$ is such that $c_{ij} = 0$ if $j = 2k + 1$, $k = N_1 - 1$ and $i = 1, ..., m$, and $c_{ij} = \nu(\psi_j^{'}(x_i^s) - \psi_j^{'}(x_i^s + s_i))$, $i = 1, ..., m$, $j = 2k$, $k = 1, ..., N_1$. Next calculating matrix Cd , where $d = diag[d_1, ..., d_{N_1}]$, and requiring that at least one entry in each column of this matrix product is distinct from zero, we get:$\forall i = 1, ...m \,\&\, j = 2k$, $k = 1, ..., N_1 | \, \exists c_{i,2k} \neq 0$. The last condition evoked one which has been used above as the controllability condition.

Theorem 4.2. *Pair (Q_1, C) is observable if condition (4.1) holds with obvious exchange in it of the lengths and locations of actuators to the corresponding sensor's parameters.*

Thus we conclude that controllability and observability conditions turn out to be the same for this system. Indeed, this is not a coincidence since a piezoelectric patch can be used as either actuator or sensor due to the dual property of piezoelectric patches.

5 Error Analysis

For the rest of the paper, $\|\cdot\|$ refers to the Euclidean norm. Accounting for bounded but unknown external forces, measurement noise, and spillover in observations jeopardize the convergence of $\|e\|$ and $\|z\|$ to zero - the ultimate goal of vibration control. However, the magnitudes of these norms can be drastically reduced by the appropriate choice of L and K. Let $Q_L = Q_1 - LC$ and $Q_K = Q_1 - BK$. Since [23]

$$e(t) = \exp(Q_L t)e(0) + \int_0^t \exp(Q_L(t-\tau))(\mathcal{F}_1 + L\varepsilon)d\tau$$

$$z(t) = \exp(Q_K t)z(0) + \int_0^t \exp(Q_K(t-\tau))(\mathcal{F}_1 + BKe)d\tau$$

we get

$$\|e\| \leq e^{-\lambda_L t}e(0) + \frac{\|\mathcal{F}_1(t)\| + \|L\|\|\varepsilon\|}{\lambda_L}(1 - e^{-\lambda_L t}) \tag{5.1}$$

$$\|z\| \leq e^{-\lambda_K t} z(0) + \frac{\|\mathcal{F}_1(t)\| + \|BK\|\|e\|}{\lambda_K}(1 - e^{-\lambda_K t}) \tag{5.2}$$

where $\lambda_L, \lambda_K > 0$ are the minimums of the absolute values of the real parts of the eigenvalues of the matrices A_L and A_K respectively. Both bounds include steady-state and transient components, and the later decay exponentially to zero. Observe that the bound for the steady-state error term in (5.1), which is called $\|e_{\rm st}\|$, does not approach zero as $\lambda_L \to \infty$ since in this case $\|L\| \to \infty$ as well. Consequently, a practical solution is to find an L which marginalizes/minimizes $\|e_{\rm st}\|$. Next we determine K marginalizing the norm of the steady-state component: $\|z_{st}\|$ in (5.2). Note that $\|z_{st}\| \nrightarrow 0$ when $\lambda_K \to \infty$ since in this case $||K|| \to \infty$ and a compromise should be reached by chosen an appropriate K. We recall also that, in the practical applications, it is commonly assumed that $\lambda_K < \lambda_L$.

6 Residual Bounds

This section estimates the contribution of uncontrolled modes to the solution of initial PDE-model, i.e. $res(w(t)) = \sum_{i=N+1}^{\infty} w_i(t)\psi_i(x)$, which is commonly known as the residual component. Time-histories of the correspondent components of the state vector are governed by mutually decoupled systems of second-order ODEs

$$\dot{W}_i = \Upsilon_i W_i + \mathit{\Gamma}_i,\ i \geq N_1 + 1 \tag{6.1}$$

where vectors: $W_i = [w_i\, \dot{w}_i]^T$ and $\mathit{\Gamma}_i = [0\, F_i]$. The last equations are independent upon control forces due to the choice of gain functions $k_i^1(x)$ and $k_i^2(x)$, and the residual component is determined by the internal model components. We adopt here a popular in these applications Kelvin-Voigt damping model which implies that that (6.1) is exponentially stable and the maximum of real parts of the eigenvalues of this system:$\lambda_{i1,2} = -(1/2)\xi(1 \pm (1 - 4/\xi)^{1/2}$, where $\xi = \alpha\sigma_i^4$, i.e.

$$\max_i(real(\lambda_{i1,2}) < -1,\ i > N_1$$

Now expending initial displacement and velocity of the beam in the series of eigenfunctions: $w^0(x) = \sum_{i=1}^{\infty} w_i(0)\psi_i(x)$ and $w^1(x) = \sum_{i=1}^{\infty} \dot{w}_i(0)\psi_i(x)$, we get that $|w_i(0)| \leq \widetilde{w}/i^2$ and $|\dot{w}_i(0)| \leq \hat{w}/i^2$, $i = 1, 2...$ since $w^0(x)$ and $w^1(x)$ are twice continuously differential by our assumption. Note that due to the same reasoning $|F_i| \leq F^*/i^2$. Then

$$W_i(t) = (exp(\Upsilon_i t)W_i(0) + \int_0^t exp(\Upsilon_i(t-\tau))F_i(t)dt$$

Next we get:

$$||W_i(t)|| \leq ||(exp(\Upsilon_i t)||||W_i(0)|| + \int_0^t ||exp(\Upsilon_i(t-\tau))||||F_i(t)||dt$$

and subsequently:

$$||W_i(t)|| \leq (1/i^2)((exp(real(\lambda_i t))W^* + F^* \int_0^t exp(real(\lambda_i(t-\tau))dt)$$

where $W^* = const$ is determined by $\widetilde{w}$ and $\hat{w}$. Due to the last inequality and since $|w_i(t)|$ and $|\dot{w}(t)_i| \leq \sqrt{2}||W_i(t)||$, we get that

$$|w_i(t)|,\ |\dot{w}_i(t)| \leq (\sqrt{2}/i^2)((W^* - F^*)e^{-t} + F^*) = \sqrt{2}\vartheta(t)/i^2,\ i > N_1$$

where $\vartheta = (W^* - F^*)e^{-t} + F^*$.

Estmation of residual term is now straightforward. Write

$$||res(w(t))|| \leq \sum_{i=N_1+1}^{\infty} ||w_i(t)||||\psi_i(x)|| \leq \sqrt{2} \sum_{i=N_1+1}^{\infty} ||w_i(t)|| \leq 2\theta \sum_{i=N_1+1}^{\infty} i^{-2}$$

Now

$$\sum_{i=1}^{\infty} i^{-2} = \pi^2/6$$

and

$$\sum_{n=1}^{N_1+1} i^{-2} = H_{N_1,2}$$

where $H_{N_1,2}$ is a generalized harmonic number. Thus, finally:

$$Res(w(t)) \leq \vartheta(t)(\pi^2/3 - 2H_{N_1,2})$$

7 Simulations

This section demonstrates in simulations that the designed output controller stabilizes buckling of the beam developing by overcritical values of the axial force as well as drastically reduces the amplitudes of resonance vibrations prompted by transverse resonance forces. Our simulations validate that the control forces applied to some critical buckling or resonance modes do not affect the dynamics of the residual infinite dimensional counterpart of the initial system. We simulate combine contributions of measurement noise and observer spillover and conclude that their impact do not alter the dynamics of the controlled system and turns out to be relatively small in practical applications.

In simulation we lift the standard assumption on smoothness of traverse force and initial functions which are embraced in our theoretical analysis.We demonstrate that the performance of this controller remains intact when this force is non-smooth and assumes finite-jumps in time and space. Smoothness of initial displacement and velocity of the beam has not been enacted in our simulations as well. We only accept that expansions of transverse forces and initial functions in the series of eigenfuctions yield series with bounded coefficients. Note that parameter values for the beam and piezoelectric patches are adopted from [7].

Figures 1- 7 present control of resonance vibrations of the beam developed by transverse forces modeling as piece-wise constant functions, i.e. products of rectangular pulses in spatial and temporal variables. The temporal part includes periodic components with frequencies equal to the first natural frequencies of the beam, while the special component is centered about midpoint of the beam and covered five percent of its length. The beam's eigenmodes corresponding to resonance frequencies of the transverse force are termed below as controlled ones. The amplitude of the transverse force is set at $1000N/m$. An axial force in this set of simulations is sub-critical and equals 1/10 of the first Euler's critical force for this beam. The measurement noise component is ignored currently, but is involved later on. Three equally spaced but non-collocated sensors and actuators are distributed over the beam. The lengths of all piezoelectric patches are equal to each other and do not exceed a percent of the beam's length. The control aims suppression of resonance vibrations of the first three primary modes without affecting the rest of modes. In the following simulations $\lambda_L = -275$ and $\lambda_K = -110$. The corresponding gain-matrices are computed using Matlab's 'place'-function.

The simulations demonstrate that the amplitudes of resonance displacements and velocities of the beam's center is suppressed about 196 and 184-times respectively. The objectives of control are attained by feedback replacing the eigenvalues of matrix Q_K into assigned positions in the stable half-part of the complex plane. This task is reached by utility of matlab 'place'-function. It follows from

(5.1) and (5.2) that in absence of measurement noise the amplitudes of steady-state vibrations are inversely proportional to the maximal real part of controlled primary eigenvalues which, in theory, let reducing of these amplitudes to arbitrary small values. In turn, extreme reduction of vibrations require appropriate control forces which, in practice, can partly be attained by rising the number and areas of piezoelectric patches.

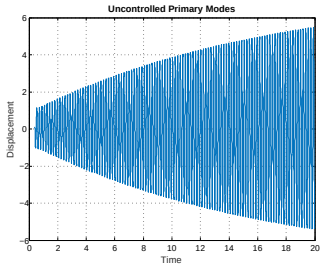

Figure 1

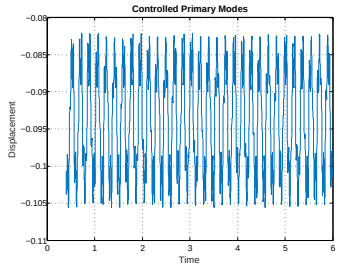

Figure 2

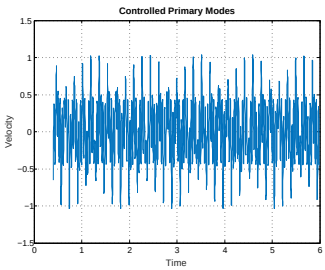

Figure 3

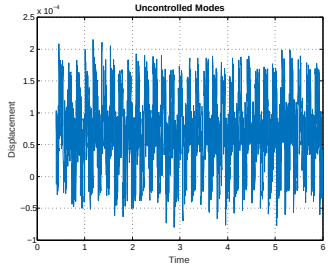

Figure 4

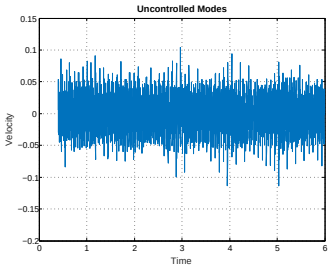

Figure 5

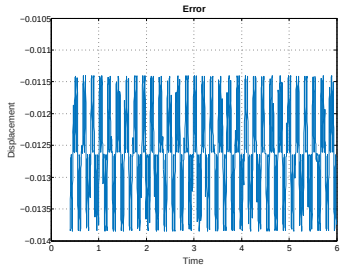

Figure 6

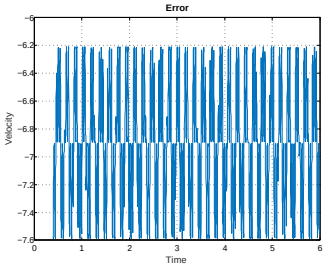

Figure 7

Figure 1 displays time-histories of displacement and velocity of the primary modes computed at the beam's center evolving in resonance vibrations with typical transient behavior. Figures 2 and 3 show that the amplitudes of these resonance vibrations are drastically suppressed by control forces. Figures 4 and 5 conform that the oscillations of uncontrolled modes are not compromised by control forces. Finally figures 6 and 7 display the time-histories of observer error-components which are mainly determined by unknown transverse forces. Note that these errors will exponentially decay to zero if the transverse forces are known.

Comparable results have been attained in controlling the beam loaded with both post-critical axial and transverse forces. Note that locations of sensors and actuators in previous and current simulations are the same. Figure 8-11 show that divergent displacement of the beam center is stabilized by applied feedback control which also reduces the amplitudes of vibrations to relatively small values. This makes accounting of geometric nonlinear components in the models of buckling control problems of the beam inessential. Small amplitude vibrations of uncontrolled naturally stable modes are unaffected by control forces in this case as well. Note that three actuators and sensors successfully stabilize buckling of the beam by an axial load which is less than the fours critical Euler's force for this beam, see figures 9 and 10 showing time-histories of vibrations of the beam's center loading by such axial force and transverse forces.

It is known that accounting for measurement noise amplifies observation error and impairs the performance of the output controller. This detriment is especially pronounced for high-gain observers magnifying the effect of noise. In our simulations we adopt a bounded noise model and set :$\varepsilon(t) = const$. Such noise primarily enlarges displacements but velocities of the controlled (primary) beam's modes. Recall that noise component in this study represents a sum of actual measurement error, which is left after filtering of the signal, and the model error retaining due to truncation of spillover components in sensor's equations. Both components are likely to be small in practice but their impact is magnified by high gains in observer and controller which should be adjusted accordingly. Our simulations show that suppression of resonance amplitudes of displacements and velocities of the beam drops to about sixty and one hundred times for $\varepsilon = 0.01m$ and 9 and 100-times for $\varepsilon = 0.1m$ respectively.

While presence of a relatively small noise component does not disrupt stabilization of the beam loading with overcritical axial force, it hampers reduction of vibration amplitudes virtually in the proportion indicating above for suppression of resonance vibrations. Thus magnitudes of measurement noise and observer spillover should be kept sufficiently small in applications of this kind of problems.

Former simulations utilize three sensors and actuators to control first three vibration modes of the bar. However, theoretically just one almost arbitrary placed sensor and actuator is sufficient for observing and controlling of a finite number of beam vibration modes.

To study how well the control intents can be attained by the minimal number of sensors and actuators, we compare the results of the above simulations with ones delivering by a single non-collocated pair of sensor and actuator. Values of λ_L and λ_k for a single and three-pair schemes of sensors and actuators have been matched in these simulations. Under this condition, we found that three pairs of evenly distributed sensors and actuators yield deeper vibration reduction than a single pair locating almost anywhere on the beam. The amplitudes of controlled resonance vibrations somewhat depend on the pair location such that the most favorable central locations yield about 77 and 143- times reduction of the amplitudes of displacement and velocity of the beam's middle-point. The previous numbers reduce to 51 and 110-times if a sensor and an actuator locate at or near beam's edges. Recall that three pairs-schema hampers respected amplitudes in 196 and 184-times. We also found that in the absence of noise one pair-schema let to put an edge of a short piezoelectric patch, serving as an actuator or a sensor, right on the edge of the beam without breaching the observability and controllability conditions of the relevant system.

Accounting for a small measurement noise somewhat impedes observability and controllability of the corresponding systems. In fact, if $\varepsilon = 0.01m$, the center of a sensor or control patches should be a bit distant from the beam edges to retain system functionality.

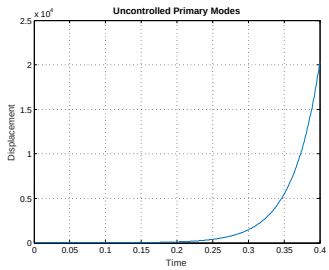

Figure 8

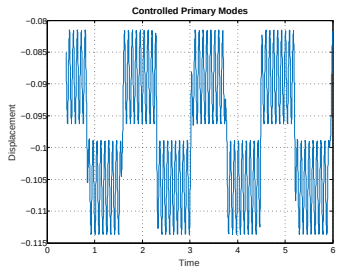

Figure 9

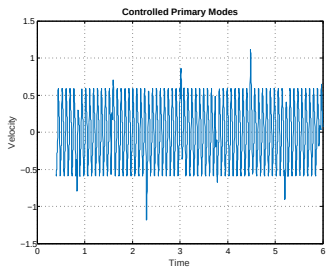

Figure 10

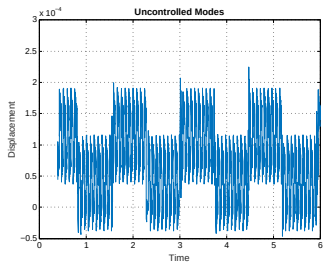

Figure 11

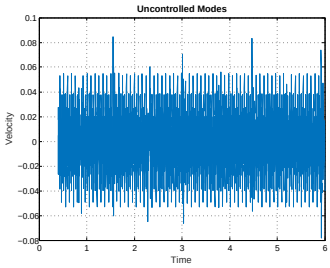

Figure 12

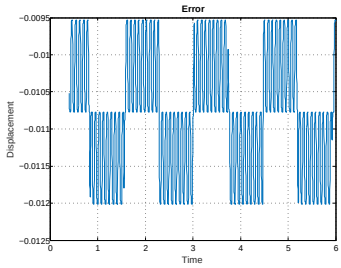

Figure 13

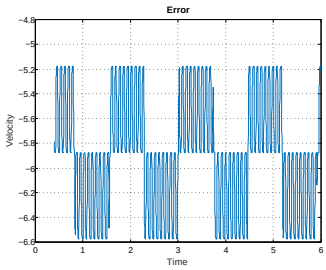

Figure 14

8 Discussions and Conclusions

This paper presents a novel approach to the design of an output controller for a typical infinite-dimensional system which is defined by linear partial differential equations. Our derivations are made for the Euler-Bernoulli model of the beam with hitched boundary conditions while they can be naturally extended to other similar models. The main benefit of this control architecture is that it voids control and downgrade observation spillovers.

We derive observability and controllability conditions and show that they turn out to be identical for the beam model, and that sensors and actuators can be placed practically anywhere on the beam except some special locations. Moreover, we show that our control objectives can be met by a pair of noncollocated sensor and actuator locating almost anywhere on the beam.

Assuming that measurements are corrupted by bounded noise and that the beam is loaded by unknown transverse forces, we estimate the norms of state and observation errors for a finite dimensional approximation of the original system which is actually controlled. These estimates illuminate impact of measurement noise, observer and controller gains and other system parameters on the bound of the state vector and can be subsequently used for optimizing controller and observer parameters.

Under standard smoothness conditions imposed on uncertain forces and initial functions, we estimate the norm of error in the solution of original system arising due to discarding the residual part in our infinite-dimensional system. Note that this derivations do not require smoothness of control forces. Our simulations validate the theoretical outcomes and demonstrate that the developed controller drastically reduces the amplitudes of resonance vibrations and efficiently stabilizes buckling

of the beam loaded by overcritical axial forces.

Due to crucial suppression of the beams steady-state dynamics, the norm of applied control force turns out to be close to the magnitude of the transverse force on the corresponding time intervals. The norm of control force increases on a short initial time interval due to peaking phenomenon.

While utility of output controllers provides valuable advantages, their application is associated with short transient behavior known as peaking phenomenon [26]. In application peaking can be mitigated by saturating the controller [26] or by utility of more adaptive piece-wise linear observers and controllers [27-29]. Relatively straightforward in this content, the implementation of these approaches will be a topic of the future research. The developed methodology is applied in this paper to a relatively simple model of the Euler-Bernoulli beam but it can be naturally extended on more complex and accurate models of elastic structures as plates, shells, etc. as well as on models of different physical essence governing by PDEs.

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