

The degree of a p -group G with $|G'| = p$

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Abstract

Let G be a finite group and $\delta(G)$ denote the least $n \in \mathbb{N}$ such that G is isomorphic to a subgroup of the symmetric group S_n . The number $\delta(G)$ is called the degree of G . This paper is a note on $\delta(G)$ for a finite p -group G with derived subgroup of order p . Our method is based on quasi-permutation representations of a finite group.

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1 Introduction

By a quasi-permutation matrix we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group G , then $\delta(G)$ is the minimal degree of a faithful permutation representation of G . Let $c(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices.

It is easy to see that

$$c(G) \leq \delta(G)$$

where G is a finite group. In this paper we determine $\delta(G)$ for some p -groups with $|G'| = p$, also we give a sharp lower bound for $\delta(G)$ in the case $Z(G) = G' \times A$ for some $A \leq Z(G)$ (Theorem 2.1.(c)).

Now, let G be a finite p -group and $|G'| = p$. By [[8], Theorem 3.1] G is isoclinic to a camina p -group H of class 2 and we have

$$\delta(H) = p^m \delta(Z(H)) = p^m \sum_{i=1}^r m_i$$

when $p > 2$ and $Z(G) \cong \prod_{i=1}^r C_{m_i}$ (See [1] and [5], Theorem 2.11).

We need some known results. Moreover let $H_G = \bigcap_{g \in G} H^g$ be the core of $H \leq G$.

Lemma 1.1 ([4], Theorem 2.2) *Let G be a finite group. Then*

$$\delta(G) = \min \left\{ \sum_{i=1}^n |G : H_i| : H_i \leq G \text{ for } i = 1, 2, \dots, n \text{ and } \bigcap_{i=1}^n (H_i)_G = 1 \right\}.$$

Lemma 1.2 ([10], Proposition 1.5) *Let G be a p -group such that $|G'| = p$ and $d(G) = 2$. Then G is minimal non-abelian.*

Lemma 1.3 ([4], Theorem 4.12 and [6]) *Let G be a finite p -group of class 2 and let $Z(G)$ be cyclic. Then*

$$\delta(G) = |G : Z(G)|^{1/2} |Z(G)|$$

if G has no Q_8 section, otherwise $\delta(G) = 2|G : Z(G)|^{1/2} |Z(G)|$.

Lemma 1.4 ([2]) *Let G be a minimal non-abelian p -group. Then*

$$\delta(G) = |G : Z(G)|^{1/2} |\delta(Z(G))|,$$

unless $G \cong Q_8$ that we have $\delta(Q_8) = 8$.

2 Main result

In this section we give our main result followed by two examples.

Let G be a finite group. Let $\mathcal{C}_i$ for $0 \leq i \leq r$ be the Galois conjugacy classes of irreducible complex characters of the group G over the rational field $\mathbb{Q}$. For $0 \leq i \leq r$, suppose that ψ_i is a representative of the class $\mathcal{C}_i$ with $\psi_0 = 1_G$. Write $\Psi_i = \sum \mathcal{C}_i$, The characters Ψ_i are called the Galois sums of G and Ψ_0 is the principal Galois sum.

Theorem 2.1 *Let G be a p -group such that $|G'| = p$. Then*

- a) *If $Z(G)$ is cyclic and G has no Q_8 section, then $\delta(G) = |G : Z(G)|^{1/2} |Z(G)|$, otherwise $\delta(G) = 2|G : Z(G)|^{1/2} |Z(G)|$.*
- b) *If $d(G) = 2$, then $\delta(G) = |G : Z(G)|^{1/2} |\delta(Z(G))|$, unless $G \cong Q_8$ that we have $\delta(Q_8) = 8$.*
- c) *If $Z(G) = G' \times A$ for some $A < Z(G)$ and for each non-principal linear character χ of A and each linear character ψ of G with $\psi_A = \chi$ holds $\Psi = |G : Z(G)|^{1/2} X$ on A (Ψ and X are the Galois sums corresponding to ψ and χ), then*

$$|G : Z(G)|^{1/2} c(Z(G)) \leq c(G).$$

Proof. Since G is nilpotent, so $[G', G] < G'$. Therefore we have $G' \leq Z(G)$ and G has nilpotency class 2. Note that for each $x, y \in G$; $[x^p, y] = [x, y]^p = 1$, thus $G^p \leq Z(G)$ and $\Phi(G) = G'G^p \leq Z(G)$. Therefore $G/Z(G)$ is elementary abelian. By [[9], Theorem 7.5 and Example 7.6] we have $[G : Z(G)] = p^{2m}$ for some positive integer m , $\text{cd}(G) = \{1, p^m\}$, and G has $|G|/p$ characters of degree 1 and $|Z(G)| - |Z(G)|/p$ irreducible characters of degree p^m . For (a) see Lemma 1.3. For (b) see Lemmas 1.2 and 1.4. Finally for (c) see [[3], Lemma 2.3] and note that $\text{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. ■

In the next Example we use Theorem 2.1.(c).

Example 2.2 *Let*

$$G = \langle x, y, z \mid x^{2^{m-2}} = 1, y^2 = z^2, xy = yx, x^z = xy, yz = zy \rangle \quad (m \geq 4).$$

Set $A = \langle x^2 \rangle$. Then since $G' = \langle y \rangle$, so $Z(G) = G' \times A$. It is straightforward to see that for each non-principal linear character χ of A and each linear character ψ of G with $\psi_A = \chi$ we have $\Psi = |G : Z(G)|^{1/2} X$ on A . Now, since $|G| = 2^m$ and $c(Z(G)) = 2^{m-3} + 2$, so $2^{m-2} + 4 \leq c(G)$ by Theorem 2.1.

Let $H = \langle x \rangle$ and $K = \{1, y, z, yz\}$. Then $H_G \cap K_G = 1$. Thus

$$c(G) \leq \delta(G) \leq |G : H| + |G : K| = 2^{m-2} + 4$$

by Lemma 1.1 and $\delta(G) = 2^{m-2} + 4$. Note that in this example we have the equality

$$|G : Z(G)|^{1/2} c(Z(G)) = c(G).$$

Example 2.3 ([7], Section 4) Let G be a non-abelian group of order p^4 and $Z(G)$ be an elementary abelian group. Also let $\Phi(G) < Z(G)$. Then $d(G) = 3$, $|G'| = p$ and $Z(G) = G' \times A$ for some $A < Z(G)$ of order p . Now, since $c(G) = p^2 + p$ and $|G : Z(G)|^{1/2} c(Z(G)) = p(p + p) = 2p^2$, so we have

$$|G : Z(G)|^{1/2} c(Z(G)) > c(G).$$

3 Conclusion

In Theorem 2.1 we determined $\delta(G)$ for some p -groups G in which $|G'| = p$. Example 2.2 shows that the bound provided in Theorem 2.1.(c) is best possible. Infact the last equality of this Example is true for our p -groups G with $Z(G) = G'$ of order p (See [1], Lemma 2.2), however, the Example 2.3 shows that this is not the case in general.

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