A Mathematical Model for Predicting the Relaxation of Creep Strains in Materials

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ABSTRACT

To describe the time dependent response of a variety of viscoelastic materials, a one-dimensional nonlinear rheological mathematical model with constant material parameters is developed by using the stress decomposition theory. The model represents, under relaxation of stress**,** the time versus deformation variation as a decay Gompertz-type function, which is able to reproduce the qualitative decay sigmoid shape of the experimental creep relaxation data of a variety of materials. Numerical applications performed shown that the model is very sensitive to material parameters variation, and particularly to the total deformation experienced by the material of interest under creep proc**ess.**

Keywords: Creep relaxation Gompertz-type model, logarithmic elastic force, ***Kelvin-Voigt model****, mathematical modeling, viscoelasticity.*

In characterization of materials, the mechanical properties are described often as purely elastic**,** plastic or viscoelastic behavior, following that the time dependent effect is neglected or taken into consideration. But, it is well known that real materials are time and history-dependent, to say, viscoelastic materials. These materials exhibit various responses to loading. Under a constant deformation, a viscoelastic material will relax and experience a decrease in stress with time. This is termed stress relaxation. On the other hand, if a viscoelastic material is subjected to a constant stress, the strain will then increase with time. This phenomenon is called viscoelastic creep. Creep response is a function of stress and time. It is usually studied in terms of strain-time curves (Alfrey and Doty, 1945; Schapery, 2000; Thompson, 2009). Viscoelastic materials manifest also a delayed recovery of deformation after the stress is removed, consisting of an elastic deformation followed by gradual decrease deformation (Schapery, 2000; Morgounov, 2001; Haslach Jr, 2005; Xia et al., 2006; Thompson, 2009; Mustalahti et al, 2010). This is creep relaxation phenomenon (Morgounov, 2001; Thompson, 2009; Mustalahti et al, 2010). If during unloading behavior the deformation is not completely recovered, the material displays then a viscoplastic response. Under cyclic loading, viscoelastic materials show a hysteresis phenomenon. This consists of a dissipation of energy through successive loading and unloading cycles. A large variety of experimental data have shown that several soft biological tissues, for example, under physiological conditions, exhibit a nonlinear sigmoidal hysteresis curve on loading and unloading (Fukushima and Homma, 1988; Thompson, 2009).Therefore, the unloading response may be used to differentiate purely elastic from viscoelastic or plastic materials. Several engineering and biomedical applications using viscoelastic materials require the formulation of time dependent deformation model. There are, in viscoelastic modeling, two categories of theory. The first is the classical linear viscoelastic theory, which is represented usually in the Boltzmann single integral form or in differential equation. This approach has been used by several investigators to describe linear viscoelastic response of materials. de Haan and Sluimer formulated a standard linear solid model including a mass for studying the dynamic behavior of building materials (de Haan and Sluimer, 2007). Chazal and Moutou Pitti (2010) using a discrete spectrum representation for the creep and relaxation differential approaches, and also a creep integral approach (2011), developed incremental constitutive relations for linear viscoelastic

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analysis. The well known established linear viscoelastic theory is, however, only valid for small deformations or low stresses (Xia et al., 2006). The second type of theory is the nonlinear viscoelastic theory which has not, contrary to the linear theory, a definitive constitutive formulation (Dealy, 2007; Ewoldt et al., 2008; 2009; Wineman, 2009). Since viscoelastic materials exhibit time dependent highly large deformations, the linear viscoelastic theory is inapplicable and then, nonlinear viscoelastic models are required. For example, it is well known in biomechanical studies that arterial tissue undergoes large deformations when it is subjected to physiological load. Thus its mechanical properties are essentially nonlinear and could not be represented on the basis of the classical linear viscoelasticity (Haslach Jr, 2005**)**. Different theoretical formulations of varying complexities have been developed for investigating the nonlinear time dependent properties of viscoelastic materials. Thus, many publications on time dependent nonlinear behavior of materials based on non-equilibrium thermodynamics (Haslach Jr, 2005; Xia et al., 2006), visco-hyperelasticity using the decomposition of the deformation gradient into elastic and viscous components (Holzapfel et al., 2002; Laiarinandrasana et al., 2003; Marvalova, 2007) and computational modeling (Weiss et al., 1995; Weiss and Gardiner, 2001) can be distinguished. Integral and differential nonlinear rheological models are also developed for characterizing various types of materials (see, e.g., for a detailed review of articles, Xia et al., 2006; Chotard-Ghodsnia and Verdier, 2007**;** Drapaca et al., 2007; Wineman, 2009). Karra and Rajagopal **(**2010) derived a generalization of the standard linear solid model**.** The model has been based on a thermodynamic framework and has been successfully applied to predict the viscoelastic response of polymide resin**.** In mechanics, the use of rheological models consisting of a combination of spring and dashpot is proved useful to describe viscoelastic behavior of materials. These rheological models are interesting, since they represent the dynamic response of materials concerned in terms of differential equations that can be solved for various particular cases of consideration (Alfrey and Doty, 1945). So much constitutive equations are derived from these combinations of spring and dashpot in order to predict and simulate material properties, and analyze experimental data. In this regard, to model materials nonlinear properties, the linear viscoelastic theory can be modified and extended to higher order stress or strain terms. A number of recent successful theoretical models have been developed on the basis of classical linear viscoelastic models extension to large deformations (Corr et al., 2001; Monsia, 2011a, 2011b, 2011c, 2011d). Corr et al. (2001) developed a nonlinear generalized Maxwell fluid model in terms of a Riccati differential equation that represents successfully the stiffening response of some viscoelastic materials. In Corr et al. (2001) and Monsia (2011a, 2011b, 2011c), only the elastic nonlinearity is taken into account by introducing a nonlinear spring force in the classical linear rheological models. Consequently, these models are insufficient to account for complete characterization of viscoelastic materials. The model (Monsia, 2011d), which is a nonlinear generalized Maxwell fluid model, taking into consideration both elastic and viscous nonlinearities, appeared useful for representing accurately the viscoelastic materials time-dependent behavior. However, these models fail to include the inertia of the mechanical system studied in the constitutive equations, in the perspective that viscoelastic materials are characterized simultaneously not just by elastic and viscous contributions, but also by an inertial function. Moreover, there are only a few theoretical models that are formulated with constant-value material coefficients so that, the material functions are considered as stress, strain or strain rate dependent. According to Haslach Jr (2005) and Xia et al. (2006) there exist only some constitutive nonlinear models capable for representing accurately the creep relaxation, to say, the unloading behavior of viscoelastic materials. The necessity to investigate the unloading behavior or creep relaxation of materials remains, even if the phenomenon is well known from many experimental data (Fukushima and Homma, 1988; Xia et al., 2006). In this regard, satisfactory nonlinear viscoelastic models are required**.** Due to material nonlinearities, a consistent constitutive equation should take then into account all together the elastic, viscous and inertial nonlinearities and relate mathematically stress, strain and their higher time derivatives (Bauer et al., 1979; Bauer, 1984). In contrast to these preceding models, the model (Monsia, 2011e) has been constructed by taking into consideration the elastic, viscous and inertial nonlinearities simultaneously. The model (Monsia, 2011e) attempted successfully to represent mathematically a complete characterization of viscoelastic materials. This model (Monsia, 2011e) was founded on the stress decomposition theory developed previously by Bauer (1984) for a complete characterization of viscoelastic arterial walls. The Bauer's theory (1984) allows, in effect, solving the mathematical complexities in rheological modeling and accounting simultaneously for high elastic, viscous and inertial nonlinearities characterizing viscoelastic materials. The Bauer's theory (1984) is derived from the classical Kelvin-Voigt model (See Figure 1: The proposed nonlinear Kelvin-Voigt model). In this theory (Bauer, 1984), the total stress acting on the material is decomposed as the sum of three components, that is, the elastic, viscous and inertial stresses. The purely elastic stress is written as a power series of strain, the purely viscous stress as a first time derivative of a similar power series of strain, and the purely inertial stress as a second time derivative of a similar power series of strain. The Bauer's stress decomposition method (1984) has been after used by many authors (Armentano et al., 1995; Gamero et al., 2001; Monsia et al., 2009) for the complete characterization of arterial behavior. In (Monsia et al., 2009), following the Bauer's approach (1984), the elastic stress is expanded in power series of strain. Monsia (2011e), using the Bauer's method (1984), developed a hyperlogistic equation that represents successfully the time-dependent mechanical properties of viscoelastic materials by expressing the elastic stress as an asymptotic expansions in powers of deformation. The viscous stress is formulated as a first time derivative of similar asymptotic expansions in powers of deformation. The inertial stress is given as a second time derivative of similar asymptotic expansions in powers of deformation. Recently, Monsia (2011f, 2012) formulated in a single differential equation the Bauer's stress decomposition theory (1984) with an exciting stress term, depending on a nonlinear elastic spring force function $\varphi(\varepsilon)$, where the scalar function $\varepsilon(t)$ represents the time dependent deformation of the mechanical system under study. In (Monsia, 2011f), the function $\varphi(\varepsilon)$ is written as a hyperbolic function, which led, in the absence of exciting stress, the author to obtain, after an adequate mathematical manipulation, a useful hyper-exponential type function representing the time versus strain variation of the viscoelastic material considered. The same author (Monsia, 2012), considering also the hyperbolic elastic spring force law $\varphi(\varepsilon)$, with now the presence of a constant exciting stress, developed successfully, after consistent mathematical operations, a nonlinear mechanical model applicable for representing the nonlinear creep behavior of viscoelastic materials. More recently, in Monsia and Kpomahou (2012), the authors, by using the Bauer's theory as formulated previously in Monsia (2011f, 2012), and expressing the nonlinear elastic spring force function $\varphi(\varepsilon)$ in a Newton's binomial function, constructed successfully a four-parameter mechanical model to represent the dynamic response of viscoelastic materials. In Monsia and Kpomahou (2012), the binomial law exponent controlled the material model nonlinearity. Numerical applications performed by the authors (Monsia and Kpomahou, 2012), clearly showed the powerful predictive ability of the model to reproduce any S-shaped experimental data. These studies demonstrate the authoritative suitability of the Bauer's stress decomposition theory (1984) as an advanced mathematical tool in rheological modeling. The use of the Bauer's theory (1984) requires overcoming two major difficulties. The first consists of a suitable choice of the nonlinear elastic force function $\varphi(\varepsilon)$ that should tend towards the expected linear hookean behavior for small deformations. The second difficulty results in the fact that the application of the Bauer's theory (1984) leads often to solve a Liénard second order nonlinear ordinary differential equation that is generally non-integrable. These considerations show that the use of the Bauer's theory (1984) to model the material nonlinear time dependent properties is not a simple task. In this paper, we have considered also the Monsia formulation (2011f, 2012) of the Bauer's approach (1984). From this approach, a one-dimensional nonlinear rheological model with constant material parameters that includes elastic, viscous and inertial nonlinearities simultaneously, is developed. The model permitted to describe accurately the unloading response of a viscoelastic material assumed to be primarily subjected to constant loading, by using a logarithmic elastic spring force law

$$\varphi(\varepsilon) = \ln(\varepsilon_o - \frac{\varepsilon}{\varepsilon_o}) \qquad \textbf{(1)}$$

where $\varepsilon_o \neq 0$, is a material constant, and $\ln$ denotes the natural logarithm.

This function is defined if and only if $\varepsilon_o > \sqrt{\varepsilon}$. $\varphi(\varepsilon)$ has a vertical asymptote at $\varepsilon = \varepsilon_o^2$, so $\varphi(\varepsilon)$ is not defined after the value ε_o^2. Therefore, the current law $\varphi(\varepsilon)$ has the advantage, contrary to previous nonlinear elastic functions used in Monsia (2011f, 2012) and, Monsia and Kpomahou (2012), to limit the magnitude of the strain $\varepsilon(t)$ by a scaling factor ε_o. The use of this law allowed deriving the time dependent deformation relationship as a decaying Gompertz-type model that reproduces successfully the qualitative S-shaped curve of the experimental creep relaxation data mentioned in many publications (van Loon et al., 1977; Chien et al., 1978; Fukushima and Homma, 1988; Morgounov, 2001; Haslach Jr, 2005; Xia et al., 2006; Thompson, 2009; Mustalahti et al., 2010). Numerical studies performed allowed also investigating the effects of rheological coefficients variation on the model.

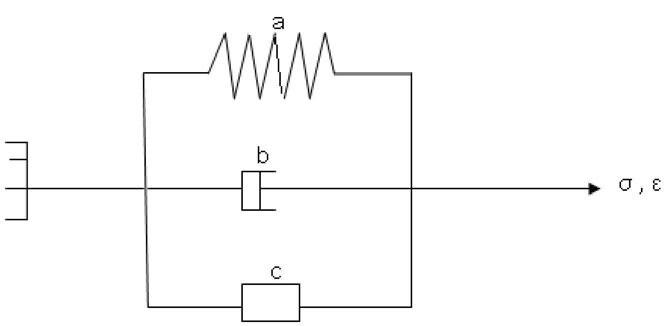

Fig.1. The proposed nonlinear rheological model

2. FORMULATION OF THE MECHANICAL MODEL

2.1 Theoretical Considerations

The present part is devoted to describe the governing equations of the theoretical model including the nonlinear elastic, viscous and inertial contributions characterizing viscoelastic materials. As pointed out previously in Monsia (2011f, 2012), the nonlinear ordinary differential equation resulting from the use of the Bauer's theory (1984), by superposing the pure elastic, viscous and inertial stresses, for a nonlinear elastic spring force function $\varphi(\varepsilon)$, can be written in the form

$$\ddot{\varepsilon}\frac{d\varphi}{d\varepsilon}+\dot{\varepsilon}^2\frac{d^2\varphi}{d\varepsilon^2}+\frac{b}{c}\dot{\varepsilon}\frac{d\varphi}{d\varepsilon}+\frac{a}{c}\varphi(\varepsilon)=\frac{1}{c}\sigma_t \qquad (2)$$

The dot over a symbol denotes a differentiation with respect to time t. The inertial module c is different from zero and time independent. The parameters a and b are respectively the stiffness and viscosity coefficients. They are also time independent material parameters. σ_t, which is a scalar function, means the total exciting stress acting on the mechanical system studied. It is required, in order to progress in the present modeling, to identify the nonlinear elastic force function $\varphi(\varepsilon)$ of interest. As stated earlier, the function $\varphi(\varepsilon)$ should obey to the basic principle governing the Bauer's theory, that is to say, behave linearly as the classical hookean elastic spring force function, for small values of deformation $\varepsilon(t)$. Following this principle, in the present study, the nonlinear elastic force function $\varphi(\varepsilon)$ is expressed in terms of a logarithmic function given by Equation (1). By using Equation (1), Equation (2) becomes

$$\sigma_t=-c\frac{\ddot{\varepsilon}(\varepsilon_o^2-\varepsilon)+\dot{\varepsilon}^2}{(\varepsilon_o^2-\varepsilon)^2}-b\frac{\dot{\varepsilon}}{\varepsilon_o^2-\varepsilon}+a\ln(\varepsilon_o-\frac{\varepsilon}{\varepsilon_o}) \qquad (3)$$

Equation (3) shows mathematically in the single differential form the constitutive relation between the total exciting stress σ_t and the resulting strain $\varepsilon(t)$. Equation (3) represents a second order nonlinear ordinary differential equation in $\varepsilon(t)$ for a given exciting stress σ_t.

2.2 Solution using an exciting stress $\sigma_t=0$

2.2.1 Evolution Equation of Deformation $\varepsilon(t)$

In the absence of exciting stress ($\sigma_t=0$), to say, in the relaxation phase where the applied stress in the creep test is removed**,** the internal dynamics of the mechanical system under study is governed by the following nonlinear ordinary differential equation

$$c\ddot{\varepsilon}(\varepsilon_o^2-\varepsilon)+\dot{\varepsilon}(c\dot{\varepsilon}+b(\varepsilon_o^2-\varepsilon))-a(\varepsilon_o^2-\varepsilon)^2\ln(\frac{\varepsilon_o^2-\varepsilon}{\varepsilon_o})=0 \qquad (4)$$

Equation (4) represents analytically the nonlinear evolution equation of deformation $\varepsilon(t)$ of the considered mechanical system under unloading.

2.2.2 Solving Time-Deformation Equation

For solving Equation (4), a change of variable is needed. Making the following suitable substitution

$$\exp(x)=\frac{\varepsilon_o^2-\varepsilon}{\varepsilon_o} \qquad (5)$$

Equation (4) transforms, after a few algebraic operations, in the form

$$\ddot{x}+\lambda\dot{x}+\omega_o^2x=0 \qquad (6)$$

where

$\lambda=\frac{b}{c}$**,** and $\omega_o^2=\frac{a}{c}$**.**

Equation (6) is the well-known second-order linear ordinary differential equation which describes a damped harmonic oscillator motion. The solution of Equation (6) depends on the relative magnitudes of λ^2 and ω_o^2, that determine whether the roots of characteristic equation associated with Equation (6) are real or complex numbers. Therefore, three particular cases may be studied.

2.2.2.1 Case A: $\lambda>2\omega_o$

If the damping is relatively large, that is to say, $\lambda>2\omega_o$, the roots of the characteristic equation are real quantities, and the oscillator is said to be overdamped. Thus, the mechanical system dissipates the energy by the damping force and the motion will not be oscillatory. The amplitude of the vibration will decay exponentially with time. In this particular case, integration of Equation (6) yields for $x(t)$ the following solution

$$x(t)=A_1\exp(r_1t)+A_2\exp(r_2t) \qquad (7)$$

where

$$r_1 = -\frac{\lambda}{2}(1+\delta)$$

and

$$r_2 = -\frac{\lambda}{2}(1-\delta)$$

are the two negative real roots of the characteristic equation

$$r^2 + \lambda r + \omega_o^2 = 0$$

with

$$\delta = \sqrt{1 - 4\frac{\omega_o^2}{\lambda^2}}$$

A_1 and A_2 are two integration constants determined by the initial conditions. Thus, using the following suitable initial-boundary conditions that account for the past history of deformation

$t \le 0$, $\varepsilon(t) = \varepsilon_i$; $t \le 0, \dot{\varepsilon}(t) = 0$

and

$t \to +\infty, \varepsilon(t) = K$

and also taking into consideration Equation (5), one can obtain the following explicit analytical solution

$$\varepsilon(t) = K + \varepsilon_o\left[1 - \exp\left[\ln(\frac{\varepsilon_o^2 - \varepsilon_i}{\varepsilon_o})(\frac{1+\delta}{2\delta}\exp(-\frac{\lambda}{2}(1-\delta)t) - \frac{1-\delta}{2\delta}\exp(-\frac{\lambda}{2}(1+\delta)t))\right]\right] \quad (8)$$

with $K = \varepsilon_o^2 - \varepsilon_o$

The first order derivative with respect to time of Equation (8) can be written

$$\dot{\varepsilon}(t) = \varepsilon_o\lambda(\frac{1-\delta^2}{2\delta})\ln(\frac{\varepsilon_o^2 - \varepsilon_i}{\varepsilon_o})\exp(-\frac{\lambda}{2}t)\sinh(\frac{\lambda\delta}{2}t)\exp\left[\ln(\frac{\varepsilon_o^2 - \varepsilon_i}{\varepsilon_o})\left[\frac{1+\delta}{2\delta}\exp(-\frac{\lambda}{2}(1-\delta)t) - \frac{1-\delta}{2\delta}\exp(-\frac{\lambda}{2}(1+\delta)t)\right]\right] \quad (9)$$

Equation (8) gives the strain versus time relationship of the viscoelastic material studied under unloading behavior. It predicts mathematically the time dependent deformation response of the material studied for some values of K as a decaying Gompertz-type model that is useful for representing an asymmetric sigmoid curve.

2.2.2.2 Case B: $\lambda = 2\omega_o$

For $\lambda = 2\omega_o$, the oscillator is said to be critically damped and the amplitude of the vibration will decay without sinusoidal oscillations during the time. In this case, Equation (6) has the solution of the form

$$x(t) = (B_1 t + B_2)\exp(-\frac{\lambda}{2}t) \quad (10)$$

where B_1 and B_2 are two integration constants determined by the initial conditions. Therefore, for the loading program

$t \le 0, \varepsilon(t) = \varepsilon_i$; $t \le 0, \dot{\varepsilon}(t) = 0$

and

$t \to +\infty, \varepsilon(t) = K$

and considering also Equation (5), the desired solution $\varepsilon(t)$ in the stress relaxation phase may be written in the following form

$$\varepsilon(t) = K + \varepsilon_o\left[1 - \exp\left[\ln(\frac{\varepsilon_o^2 - \varepsilon_i}{\varepsilon_o})(\frac{\lambda}{2}t + 1)\exp(-\frac{\lambda}{2}t)\right]\right] \quad (11)$$

where $K = \varepsilon_o^2 - \varepsilon_o$

Equation (11) describes also the strain time relationship for some values of K as a decaying Gompertz-type function adequate to fit the asymmetric S-shaped experimental data. The time derivative of Equation (11) of first order is given by

$$\dot{\varepsilon}(t) = \frac{\lambda^2}{4}\varepsilon_o \ln(\frac{\varepsilon_o^2 - \varepsilon_i}{\varepsilon_o}) t \exp(-\frac{\lambda}{2}t) \exp\left[\ln(\frac{\varepsilon_o^2 - \varepsilon_i}{\varepsilon_o})(1 + \frac{\lambda}{2}t)\exp(-\frac{\lambda}{2}t)\right] \quad (12)$$

2.2.2.3 Case C: $\lambda < 2\omega_o$

For a relatively small damping, to say, $\lambda < 2\omega_o$, the roots of the characteristic equation are complex numbers, and the oscillator is said to be underdamped. The amplitude of the vibration decreases exponentially with time. In this particular case, integration of Equation (6) yields for $x(t)$ the following solution

$$x(t) = C\exp(-\frac{\lambda}{2}t)\cos(\omega t - \phi) \quad (13)$$

where

$$\omega = \sqrt{\omega_o^2 - \frac{\lambda^2}{4}}$$

and C and ϕ are two integration constants determined by the initial conditions. Then, setting the suitable initial-boundary conditions taking into account the past history of deformation

$t \le 0, \varepsilon(t) = \varepsilon_i$; $t \le 0, \dot{\varepsilon}(t) = 0$

and

$t \to +\infty, \varepsilon(t) = K$

and taking also into consideration Equation (5), the following explicit analytical solution for the desired strain $\varepsilon(t)$ in the stress relaxation phase can be obtained

$$\varepsilon(t) = K + \varepsilon_o\left[1 - \exp\left[\ln(\frac{\varepsilon_o^2 - \varepsilon_i}{\varepsilon_o})(\cos(\omega t) + \frac{\lambda}{2\omega}\sin(\omega t))\exp(-\frac{\lambda}{2}t)\right]\right] \quad (14)$$

where $K = \varepsilon_o^2 - \varepsilon_o$

The exponentiated exponential Equation **(14)** is of the form of a Gompertz-type model in which the constant parameter $\ln(\frac{\varepsilon_o^2 - \varepsilon_i}{\varepsilon_o})$ is modulated by the sinusoidal function $\cos(\omega t) + \frac{\lambda}{2\omega}\sin(\omega t)$, and appears very useful for the asymmetric S-shaped experimental data fitting. For some values of the asymptotic parameter K, the strain $\varepsilon(t)$ will decay exponentially. The first order time derivative of deformation may be expressed as

$$\dot{\varepsilon}(t) = \varepsilon_o\omega(1 + \frac{\lambda^2}{4\omega^2})\ln(\frac{\varepsilon_o^2 - \varepsilon_i}{\varepsilon_o})\sin(\omega t)\exp(-\frac{\lambda}{2}t)\exp\left[\ln(\frac{\varepsilon_o^2 - \varepsilon_i}{\varepsilon_o})(\cos(\omega t) + \frac{\lambda}{2\omega}\sin(\omega t))\exp(-\frac{\lambda}{2}t)\right] \quad (15)$$

3. NUMERICAL RESULTS AND DISCUSSION

This section presents some numerical examples to investigate the predictive capability of the model to reproduce the mechanical response of the material considered under relaxation of stress. The dependence of strain versus time curve on the material parameters is also discussed. In the following, the material response is investigated at the fixed value $K = 0$, $\varepsilon_o = 1$. Therefore, ε_i must be less than 1.

3.1 Case A: $\lambda > 2\omega_o$

Figure 2 illustrates the typical time dependent strain behavior of viscoelastic materials studied, resulting from Equation (8) with the fixed value of coefficients at $\varepsilon_i = 0.9$, $\lambda = 2$, $\omega_o = 0.5$. We note that the strain $\varepsilon(t)$ decreases until the asymptotical value $K = 0$, that is, to the time-axis with increase time t, and the elastic spring force $\varphi(\varepsilon)$ becomes then equal to zero. The strain versus time curve is nonlinear, with a nonlinear beginning initial portion. Thus, Equation (8) reproduces the qualitative decay S-shape of the experimental unloading data mentioned by several authors for a variety of materials (van Loon et al., 1977; Chien et al., 1978; Fukushima and Homma, 1988; Morgounov, 2001; Haslach Jr, 2005; Xia et al., 2006; Thompson, 2009; Mustalahti et al., 2010).

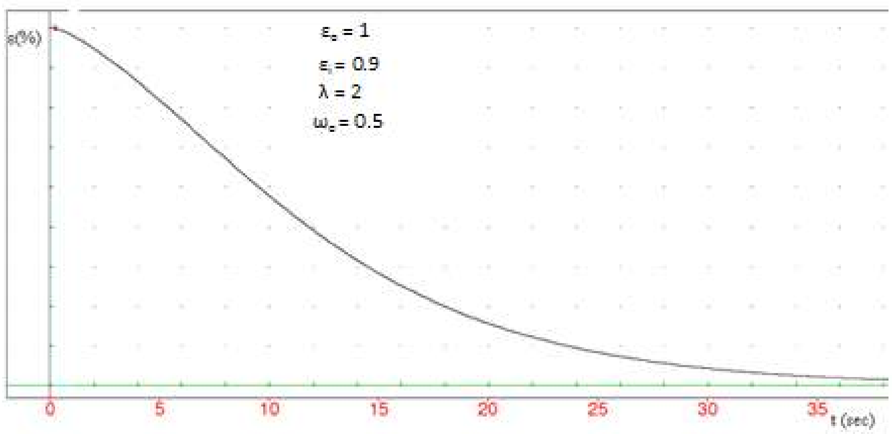

Fig.2. Typical strain versus time curve exhibiting a decay sigmoid behavior.

Figure 3 shows the comparison of the curves of strain $q(t) = \varepsilon_i - \varepsilon(t)$, and its first order time derivative $\frac{dq(t)}{dt}$, that is to say, the strain rate, obtained from equations (8) and (9). The strain rate, after reaching its peak value at the inflexion point of the strain curve, declines gradually to zero with time t, when the strain attains the strain value of the failure point. This behavior of the strain rate has been observed in (Morgounov, 2001). The strain curve $q(t)$ reproduces also the qualitative S-shaped curve derived from experiments by Lesecq et al. (1997). Recently, Mensah et al. (2009), in their theoretical work on the soft biological materials, have obtained the same S-shaped behavior of the time dependent deformation. The values of coefficients are $\varepsilon_i = 0.9$ $\lambda = 2$, $\omega_o = 0.5$.

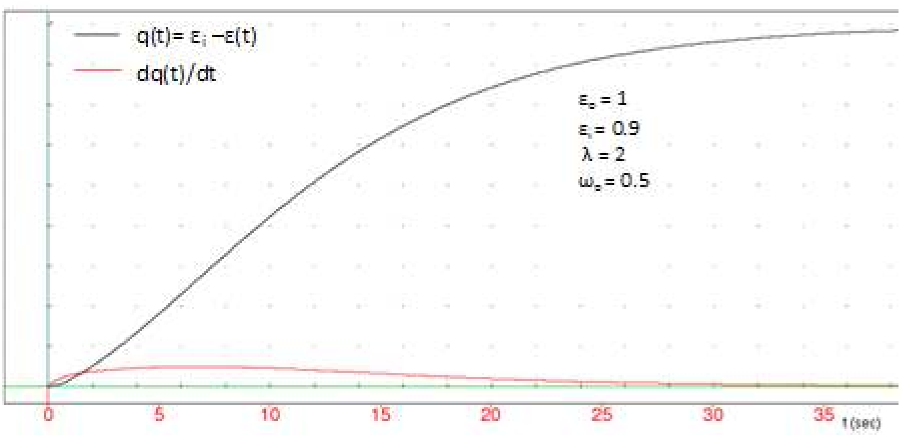

Fig.3. Comparison of time dependent curves of strain and strain rate derived from equations (8) and (9), respectively.

Figure 4, 5 and 6, illustrates the effects of material coefficients on the strain versus time curve generated by Equation (8). The effects of the action of these coefficients are studied with the help of an own computer program by varying step by step one coefficient while the other two are kept constant.

As shown in Figure 4, an increase of the viscosity coefficient λ, decreases the value of the strain on the time period considered. The slope also decreases with increase λ. The red line corresponds to $\lambda = 2$, the blue line to $\lambda = 3$, and the green line to $\lambda = 4$. The other parameters are $\varepsilon_i = 0.9$ $\omega_o = 0.5$.

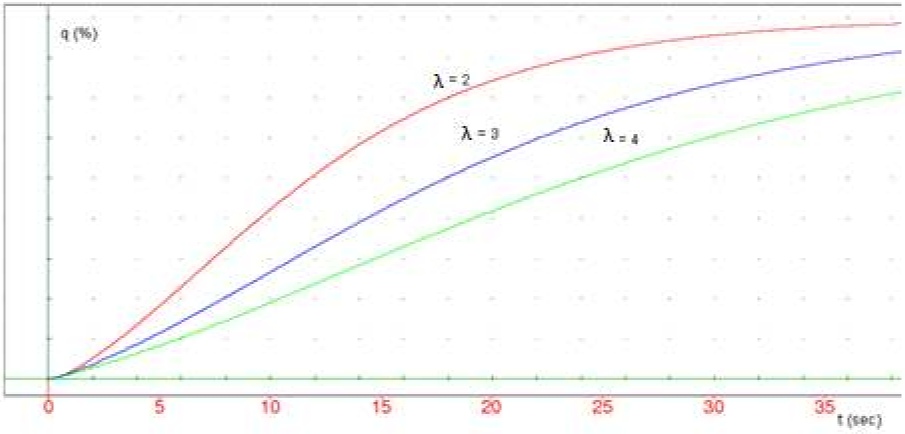

Fig. 4. Comparison of strain versus time curves for three values of the viscosity coefficient λ.

Figure 5 shows the effect of the natural frequency ω_o variation on the strain-time response. An increase ω_o, increases the strain value on the time period considered, increases also the slope and the curves become more nonlinear. The red line corresponds to $\omega_o = 0.05$, the blue line to $\omega_o = 0.1$, and the green line to $\omega_o = 0.5$. The other parameters are $\varepsilon_i = 0.9$, $\lambda = 2$.

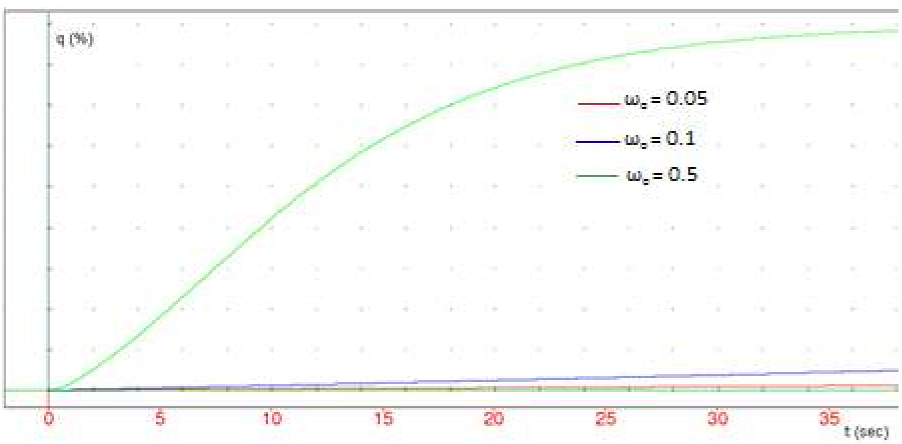

Fig. 5. Comparison of strain-time curves with three different values of the natural frequency ω_o.

From Figure 6, we note that change of the coefficient ε_i has a high effect on the peak asymptotical value of strain. We observe that an increase ε_i, increases significantly and fast the maximum asymptotical value of strain on the time period considered. The slope increases also with increase ε_i. The red line corresponds to $\varepsilon_i = 0.3$, the blue line to $\varepsilon_i = 0.6$, and the green line to $\varepsilon_i = 0.9$. The other parameters are $\lambda = 2$, $\omega_o = 0.5$.

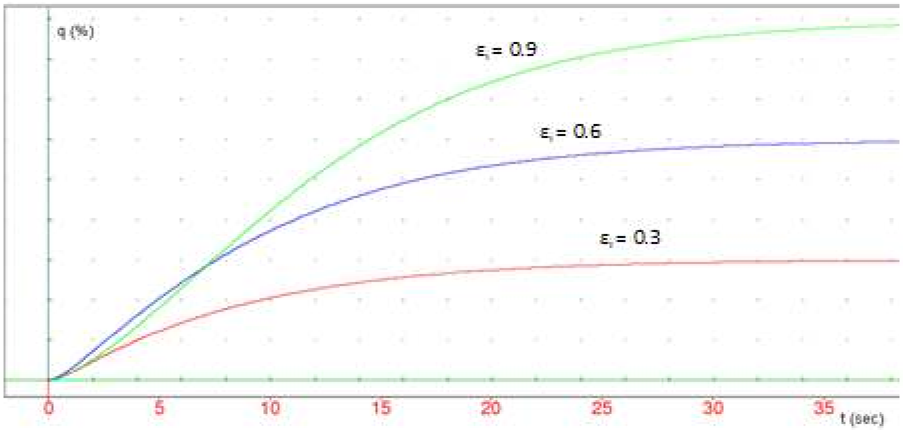

Fig. 6. Comparison of strain versus time curves showing the effect of the coefficient ε_i.

3.2 Case B: $\lambda = 2\omega_o$

Figure 7 illustrates the typical strain versus time curve derived from Equation (11), with the fixed value of coefficients at $\varepsilon_i = 0.9$, $\lambda = 1$. The curve exhibits the same limiting value of $K = 0$. The curve also shows a nonlinear decay sigmoid behavior of materials of interest as in the preceding case A.

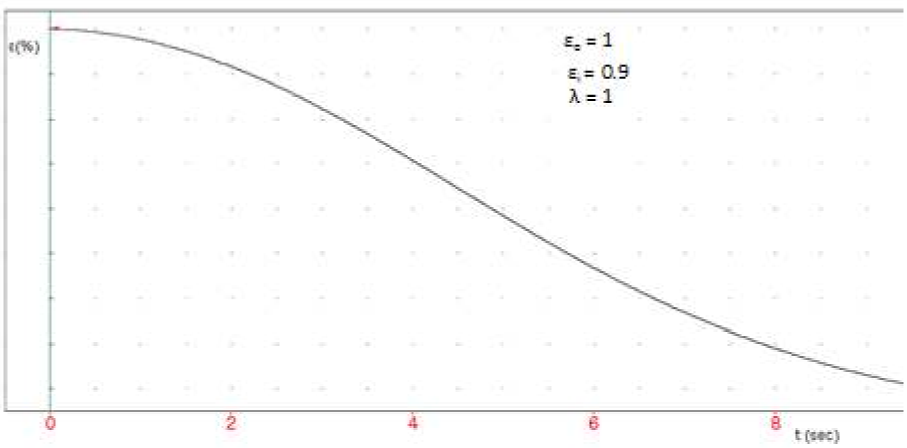

Fig.7. Typical strain time curve showing decay S-shaped behavior derived from Equation (11).

Figure 8 illustrates the comparison of the curves of strain $q(t) = \varepsilon_i - \varepsilon(t)$, and the strain rate $\frac{dq(t)}{dt}$ derived from equations (11) and (12). The strain rate, after attaining its maximum value at the inflexion point of the strain curve, reduces gradually to zero with time t, when the strain reaches the strain value of the failure point. These curves reproduce the qualitative behavior of the time dependent strain and strain rate derived by Morgounov (2001) under relaxation of stress. The current strain curve $q(t)$ reproduces also the qualitative S-shaped curve derived from experiments by Lesecq et al. (1997). The values of coefficients are $\varepsilon_i = 0.9$, $\lambda = 1$.

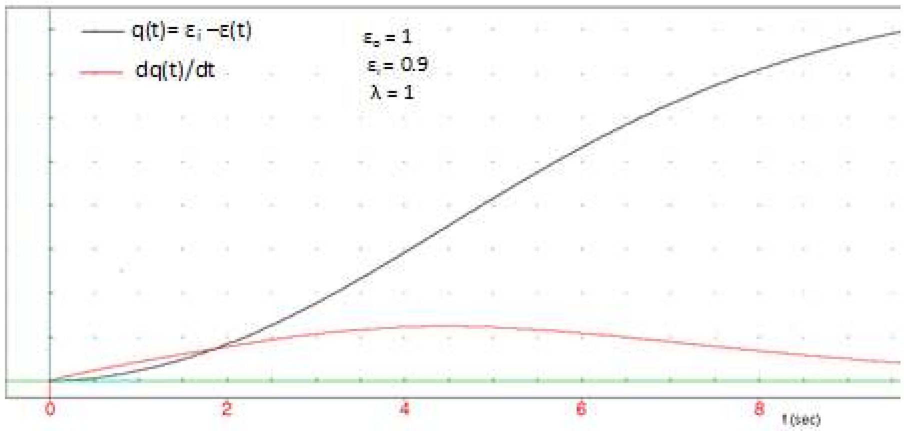

Fig.8. Comparison of time dependent curves of strain and strain rate derived from equations (11) and (12), respectively.

3.3 Case C: $\lambda < 2\omega_o$

Figure 9 demonstrates the typical strain versus time curves derived from Equation (14), with the fixed value of coefficients at $\varepsilon_i = 0.5$, $\lambda = 1$, $\omega_o = 1$ The curve exhibits the same limiting value of $K = 0$, and shows a nonlinear decay exponential behavior of materials of interest on the time period considered.

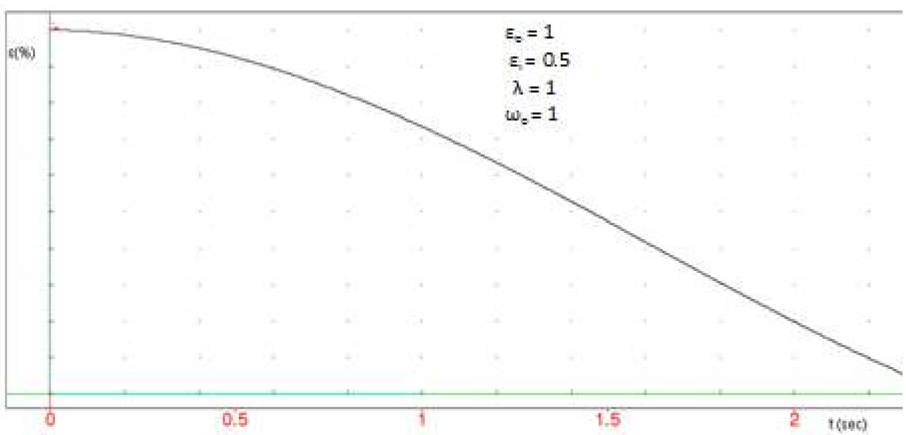

Fig.9. Typical strain time curve showing a decay exponential behavior derived from Equation (14).

Figure 10 shows the comparison of the curves of strain $q(t) = \varepsilon_i - \varepsilon(t)$, and the strain rate $\dfrac{dq(t)}{dt}$ derived from equations (14) and (15). As noticed previously, the strain rate, after reaching its maximum value at the inflexion point of the strain curve, decreases gradually to zero with time t, when the strain reaches the strain value of the failure point. These curves reproduce also the qualitative behavior of the time dependent strain and strain rate derived by Morgounov (2001) under relaxation of stress. The current strain curve $q(t)$ reproduces likewise the qualitative S-shaped curve derived from experiments by Lesecq et al. (1997) and obtained theoretically by Mensah et al. (2009). The values of coefficients are $\varepsilon_i = 0.5$, $\lambda = 1$, $\omega_o = 1$.

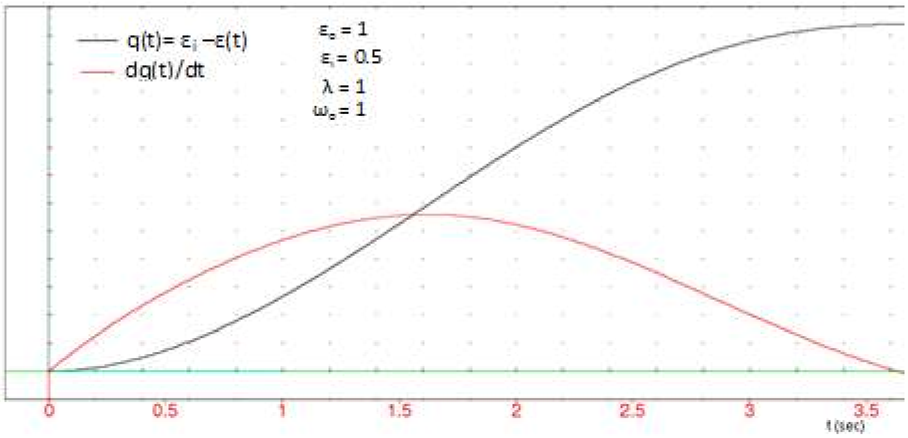

Fig.10. Comparison of time dependent curves of strain and strain rate derived from equations (14) and (15), respectively.

The preceding numerical examples demonstrated that the model is well-suited to represent the S-shaped deformation response of viscoelastic materials under unloading. The model is based on the Bauer's theory (1984) consisting to superpose the elastic, viscous and inertial nonlinear contributions for obtaining the total stress acting on the material. This method permitted to perform a complete characterization of the viscoelastic material under study. In this model, the nonlinear elastic force function is assumed to be a logarithmic law, which allowed taking into account elastic, viscous and inertial nonlinearities simultaneously, and deriving successfully the time dependent response of the material studied as a Gompertz-type function that is well known useful for reproducing an asymmetric sigmoid curve. It is also interesting to note that the Gompertz model is an asymmetric function widely used to represent increases in several growth phenomena exhibiting a sigmoid pattern, for example, in physics, biology and biomedical science. The empirical choice of the nonlinear logarithmic elastic force function $\varphi(\varepsilon)$ is inspired by the work (Covács et al., 2001) and also justified by the fact that for $\varepsilon << \varepsilon_o$, the function $\varphi(\varepsilon)$ can be developed in power series of deformation ε. In this regard, the choice of function $\varphi(\varepsilon)$ agrees very well with the polynomial function of deformation utilized by Bauer (1984) so that, for small values of deformation, $\varphi(\varepsilon)$ behaves linearly as expected. It is worth noting that the effects of variation of the natural frequency ω_o and the viscosity λ on the current model are in opposite direction. Moreover, choosing $K = 0$, means that there were almost no residual deformations even at large stress levels. This involves then almost complete recovery and, the material of interest behaves viscoelastically. In contrast to this, the coefficient K can be chosen different from zero and then, the material will behave viscoplastically. The present model can therefore, following the value of K, describes successfully the viscoelastic or viscoplastic behavior of some materials.

4. CONCLUSION

A mathematical rheological model has been developed by using the stress decomposition theory. The nonlinear elastic, viscous and inertial contributions characterizing viscoelastic materials are simultaneously taking into consideration through the use of a logarithm law for the nonlinear elastic spring force function in the present model. The time dependent deformation of a variety of materials has been investigated under creep relaxation. It has been found that the strain reduces gradually following a decay sigmoid behavior, in concordance with the experimental creep relaxation data existing in the literature. It has been found also that the total deformation under creep process has a high effect on the value of the deformation under unloading and, the natural frequency and the viscosity coefficients effect acting on the material of interest are in opposite direction. It is even observed that the increase values of material parameters, to say, of the natural frequency, the viscosity coefficient and the initial deformation, increases the nonlinear viscoelastic sensitivity. It is worth mentioning that the present model offers the ability to describe the viscoelastic behavior of the material under study as far as its viscoplastic response.

REFERENCES

Alfrey, T., Doty, P. (1945). The methods of specifying the properties of viscoelastic materials. Journal of Applied Physics, 16, 700-713. http://dx.doi.org/101063/1.1707524

Armentano, R. L., Barra, J. G., Levenson, J., Simon, A., Pichel, R. H. (1995). Arterial wall mechanics in conscious dogs. Assessment of viscous, inertial, and elastic moduli to characterize aortic wall behaviour. Circ Res., 76, 468-478.

Bauer, R. D., Busse, R., Schabert, A., Summa, Y., Wetterer, E. (1979). Separate determination of the pulsatile elastic and viscous forces developed in the arterial wall in vivo. Pflügers. Arch., 380, 221-226. http://dx.doi.org/10.1007/BF00582900

Bauer, R. D. (1984). Rheological approaches of arteries. Biorheology. Suppl., I, 159-167

Chazal, C., Pitti, M. R. (2009). Viscoelastic incremental formulation using creep and relaxation differential approches. Mech Time-Depend Mater, 14, 173-190. http://dx.doi.org/10.1007/s11043-009-9101-1

Chazal, C., Pitti, M. R. (2011). Incremental constitutive formulation for time dependent materials: creep integral approach. Mech Time-Depend Mater, 15, 239-253. http://dx.doi.org/10.1007/s11043-011-9135-z

Chien, S., Sung, K. P., Skalak, R., Usami, S. (1978). Theoretical and experimental studies on viscoelastic properties of erythrocyte membrane. Biophys. J., 24, 463-487. http://dx.doi.org/10.1016/S0006-3495(78)85395-8

Chotard-Ghodsnia, R., Verdier, C. (2007). Rheology of Living Materials, in: Mollica, F., Preziosi, L., Rajagopal, K.R. (Eds). Modeling of biological Materials. pp. 1-31 http://dx.doi.org/10.1007/978-0-8176-4411-6_1

Corr, D. T., Starr, M. J., Vanderby, Jr. R., Best, T. M. (2001). A Nonlinear Generalized Maxwell Fluid Model for Viscoelastic Materials. Journal of applied Mechanics,68,787-790. http://dx.doi.org/10.1115/1.1388615

Covács, S.J., Mc Queen, D.M., Peskin, C.S. (2001). Modelling Cardiac Fluid Dynamics and Diastolic Function. Phil. Trans. R. Soc. Lond. A., 359, 1299-1314. http://dx.doi.org/10.1098/rsta.2001.0832

de Haan, Y. M., Sluimer, G. M. (2001). Standard linear solid model for dynamic and time dependent behaviour of building materials. Heron, 46, 49-76.

Drapaca, C. S., Sivaloganathan, S., Tenti, G. (2007). Nolinear Constitutive Laws in Viscoelasticity. Mathematics and Mechanics of Solids, 12, 475-501. http://dx.doi.org/10.1177/1081286506062450

Ewoldt, R. H., Hosoi, A. E., Mckinley, G. H. (2009). Nonlinear viscoelastic biomaterials: meaningful characterization and engineering inspiration. Integrative and Comparative Biology, 49, 40-50. http://dx.doi.org/10.1093/icb/icp010

Ewoldt, R. H., Hosoi, A. E., Mckinley, G. H. (2008). New measures for characterizinq nonlinear viscoelasticity in large amplitude oscillatory shear. Society of Rheology, 52, 1427-1458. http://dx.doi.org/10.1122/1.2970095

Fukushima, T., Homma, T. (1988). A logistic-type curve fits pressure-diameter relationship for relaxed and contracted dog renal arteries. Biorheology, 25, 37-48.

Gamero, L. G., Armentano, R. L., Barra, J. G., Simon, A., Levenson, J. (2001). Identification of Arterial Wall Dynamics in Conscious Dogs. Exp. Physiol., 86, 519-528. http://dx.doi.org/10.1113/eph8602172

Haslach Jr, H. W. (2004). Nonlinear viscoelastic, thermodynamically consistent, models for biological soft tissue. Biomechan Model Mechanobiol, 3, 172-189. http://dx.doi.org/10.1007/s10237-004-0055-6

Holzapfel, G. A., Gasser, T. C., Stadler, M. (2002). A structural model for the viscoelastic behavior of arterial walls: continuum formulation and finite element analysis. Eur. J. Mech. A/Solids, 21, 441-463

Karra, S., Rajagopal, K. R. (2010). Modeling the non-linear viscoelastic response of high temperature polyimides. Preprint submitted to Mechanics of Materials, 1-16.

Laiarinandrasana, L., Piques, R., Robisson, A. (2003). Visco-hyperelastic model with internal state variable coupled with discontinuous damage concept under total Lagrangian formulation. International Journal of Plasticity, 19, 977-1000. http://dx.doi.org/10.1016/S0749-6419(02)00089-X

Lesecq, S., Baudin, V., Kister, J., Poyart, C., Pagnier, J. (1997). Influence of the A Helix Structure on the Polymerization of Hemoglobin S. Journal of Biological Chemistry, 272, 15242-15246. http://dx.doi.org/10.1074/jbc.272.24.15242

Marvalova, B. (2007). Viscoelastic properties of filled rubber. Experimental observations and material modeling. Engineering Mechanics, 14, 81-89.

Mensah, F. E., Grant, J. R., Thorpe, A., N. (2009). The evolution of the molecules of deoxy-hemoglobin S in sickle cell anaemia: A mathematical prospective. International Journal of Physical Sciences, 4, 576-583.

Monsia, M. D. (2011a). A Hyperlogistic-type Model for the Prediction of Time-dependent Nonlinear Behavior of Viscoelastic Materials. Int. J. Mech. Eng., 4, 1-4.

Monsia, M. D. (2011b). A Nonlinear Generalized Standard Solid Model for Viscoelastic Materials. Int. J. Mech. Eng., 4, 11-15.

Monsia, M. D. (2011c). A Modified Voigt Model for Nonlinear Viscoelastic Materials. Int. J. Mech. Eng., 4, 17-20.

Monsia, M. D. (2011d). A Simplified Nonlinear Generalized Maxwell Model for Predicting the Time Dependent Behavior of Viscoelastic Materials. World Journal of Mechanics, 1, 158-167. http://dx.doi.org/10.4236/wjm.2011.13021

Monsia, M. D. (2011e). Lambert and Hyperlogistic Equations Models for Viscoelastic Materials: Time-dependent Analysis. Int. J. Mech. Eng., 4, 5-10.

Monsia, M. D., Massou S., Moussiliou S. (2009). Nonlinear Mathematical Model for Viscoelastic Behavior of Arterial Wall. Rev. CAMES-Série A., 08, 12-17.

Monsia, M. D. (2011f). Modeling the Nonlinear Rheological Behavior of Materials with a Hyper-Exponential Type Function. Mechanical Engineering Research, 1, 103-109. http://dx.doi.org/10.5539/mer.v1n1p103

Monsia, M. D. (2012). A Nonlinear Mechanical Model for Predicting the Dynamics Response of Materials under a Constant Loading. Journal of Materials Science Research, 1, 90-100. http://dx.doi.org/10.5539/jmsr.v1n1p90

Monsia, M.D., Kpomahou, Y.J.F. (2012). Predicting the Dynamic Behavior of Materials with a Nonlinear Modified Voigt Model. Journal of Materials Science Research, 1, 166-173. http://dx.doi.org/10.5539/jmsr.v1n2p166

Schapery, R. A. (2000). Nonlinear viscoelastic solids. International Journal of Solids and structures,37, 359-366. http://dx.doi.org/10.1016/S0020-7683(99)00099-2

Van Loon, P., Klip, W., Bradley, E. L. (1977). Length-Force and Volume-Pressure relationships of arteries. Biorheology, 14, 181-201.

Xia, Z., Shen, X., Ellyin, F. (2006). An assessment of nonlinearity viscoelastic constitutive models for cyclic loading. The effect of a general loading /unloading rule. Mechanics of Time dependent Materials, 9, 281-300.

Weiss, J. A., Gardiner, J. C. (2001). Computational Modeling of Ligament Mechanics. Biomedical Engineering, 29, 1-70.

Weiss, J. A., Gardiner, J. C., Quapp, K. M. (1995). Material Models for the Study of Tissue Mechanics. Proceeding, International Conference on pelvic and Lower Extremity Injuries, 249-261.

Wineman, A. (2009). Nonlinear Viscoelastic Solids A Review. Mathematics and Mechanics of Solids, 14, 300-366.

http://arxiv.org/pdf/1006.5567 Mustalahti, M., Rosti, J., Koivisto, J., Alava, M.J. (2010).

http://eprints.qut.edu.au/15781/5/05Chapter4.pdf Thompson, R. E. (2009). Mechanical properties of intervertebral joints.

http://WWW.eolss.net/Sample-chapters/C06/E6-197-06-00.pdf Dealy, J. M. Nonlinear viscoelasticity. Rheology, 1