The classical mechanics from the quantum equation

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ABSTRACT

The quantum hydrodynamic analogy (QHA) is derived as the deterministic limit of its stochastic version. On large scale, the stochastic quantum hydrodynamic analogy (SQHA) shows dynamics that may acquire the classic behavior. The SQHA shows that in presence of spatially distributed noise the quantum behavior is maintained on a distance shorter than the correlation length (λ_C) of fluctuations of the modulus of the wave function. The quantum mechanics is achieved in the deterministic limit when λ_C tends to infinity with respect to the scale of the problem. Moreover, when, the physical length of the problem is of order or larger than λ_C, the model shows that the quantum potential may have a finite range of efficacy maintaining its non-local effect on a finite distance λ_L ("quantum non-locality length"). The paper also unveils that the SQHA has the corresponding stochastic Schrödinger equation as happens for the deterministic limit. In the case when the classical limit is approached, the model shows that the dynamics can be described by a non-linear stochastic Schrödinger equation at the glance with the current theoretical outputs. In particular, the work shows that the semi-empirical Gross-Pitaevskii equation can be derived by the SQHA.

Keywords: Quantum hydrodynamic analogy, quantum to classical transition; quantum decoherence; quantum dissipation; noise suppression; open quantum systems; quantum dispersive phenomena; quantum irreversibility

1. INTRODUCTION

The emergence of classical behavior from a quantum system is a problem of interest in many branches of physics. The incompatibility between the quantum and classical mechanics comes mainly from the non local character of the quantum mechanics. From the empirical point of view, it is widely accepted that fluctuations may destroy quantum coherence and elicit the emergence of the classical behavior. By using the alternative approach of the quantum hydrodynamic analogy (QHA) [1] in this paper we investigate how the fluctuations influence the quantum non locality and possibly lead to the large-scale classical evolution.

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The motivation of using the unpopular QHA relies in the fact that it owns a classical-like structure that makes it suitable for the achievement of a comprehensive understanding of quantum and classical phenomena. The suitability of the classical-like theories in explaining open quantum phenomena is a matter of fact and is confirmed by their success in the description of the dispersive effects in semiconductors, multiple tunneling, mesocopic and quantum Brownian oscillators, critical phenomena, stochastic Bose-Einstain condensation and in the theoretical regularization procedure of quantum field [2-12]. The interest for the QHA had never interrupted, resulting useful in the numerical solution of the time-dependent Schrödinger equation [13], and had led to a number of papers and textbooks bringing original contributions to the comprehension of quantum dynamics [14-16] in problem whose scale is larger that one of small atoms such as chromophore-protein complexes and semi-conducting polymers that are dynamically submitted to environmental fluctuations.

Compared to others classical-like approaches (e.g., the stochastic quantization procedure of Nelson [17] and the mechanics given by Bohm [18]) the QHA has the precious property to be exactly equivalent to the Schrödinger equation and to be free from problems such as the undefined variables of the Bohmian mechanics or the unclear relation between the statistical and the quantum fluctuations as in the Nelson theory. Concerning the last point, as clearly shown by Tsekov [19], it must be noted that the QHA has not to be confused with the Bohmian mechanics that is more like a mean-field limit of quantum mechanics.

Among the researches to which the present work has useful connections there are: The clarification of the hierarchy between the classical and quantum mechanics [17-19]; the description of mesoscopic system showing quantum-to classical transition [12]; the interplay between the fluctuations and the quantum coherence [20-24], The achievement of a consistent theory of quantum gravity [10]; The quantum treatment of chaos and irreversibility [20].

2. THEORY: THE SQHA EQUATION OF MOTION

When the noise is a stochastic function of the space, in the quantum hydrodynamic analogy the motion equation is described by the stochastic partial differential equation (SPDE) for the spatial density of number of particles n (i.e., the wave function modulus squared (WFMS)), that reads [25]

$$\partial_t \mathrm{n}_{(q,t)} = -\nabla_q \bullet (\mathrm{n}_{(q,t)} \dot{q}) + \eta_{(q,t,\Theta)} \qquad (1)$$

$$\dot{p} = -\nabla_q (V_{(q)} + V_{qu(\mathrm{n})}), \qquad (2)$$

$$\dot{q} = \frac{\nabla_q S}{m} = \frac{p}{m}, \qquad (3)$$

$$S = \int_{t_0}^{t} dt(\frac{p \bullet p}{2m} - V_{(q)} - V_{qu(\mathrm{n})}) \qquad (4)$$

where Θ is the amplitude of the spatially distributed noise η whose covariance matrix reads

$$< \eta_{(q_\alpha)}, \eta_{(q_\beta + \lambda)} > = < \eta_{(q_\alpha)}, \eta_{(q_\alpha)} > G(\lambda) \delta_{\alpha\beta} \qquad (5)$$

where $G(\lambda)$ is the dimensionless shape of the correlation function of the noise. Moreover, $V_{(q)}$ represents the Hamiltonian potential and $V_{qu(\mathrm{n})}$, the so called (non-local) quantum potential [14], that reads

$$V_{qu} = -(\frac{\hbar^2}{2m})n^{-1/2}\nabla_q \bullet \nabla_q n^{1/2}$$

Close to the quantum deterministic limit the WFMS can be developed in a series of approximation $n \cong n_0 + \Delta n_1 + \Delta n_2 + \ldots + \Delta n_n$, where n_0 is the solution of the deterministic limit [25]. The condition that the fluctuations of the quantum potential $V_{qu(n)}$ do not diverge, as Θ goes to zero (so that the deterministic limit can be warranted) is implemented by operating on the system of the discrete version of the SPDE (1) whose variable reads

$$Y_i = \iiint\limits_{\Delta_i} d^{3n}q \; n_{(q,t)}, \tag{6}$$

where the space hyper-cell $\Delta_i = [(\mathbf{q}_{(i-1)}, (\mathbf{q}_{(i)})]$ (with $\mathbf{q}_{(i)} - \mathbf{q}_{(i-1)} = \lambda$) is taken around the discrete point $q_{(\mathbf{i})}$.

In the limit of small Θ, the quantum potential fluctuations can be derived as a function of the fluctuations of the WFMS at the smallest order $n_0 + \Delta n_1$. The results show that, in order to have the quantum potential energy finite in the fluctuating state, i.e., $\lim_{\lambda \to 0} \langle V_{qu}, V_{qu} \rangle$ finite, the following conditions must be fulfilled

$$\lim_{\lambda \to 0} \sum_{\alpha} \lambda^{-2} [1 - G_{(\lambda)}] < \infty \tag{7}$$

$$\lim_{\lambda \to 0} \sum_{\alpha} \lambda^{-4} [1 - G_{(\lambda)}]^2 < \infty \tag{8}$$

$$\lim_{\lambda \to 0} \sum_{\alpha} \lambda^{-4} [3 + G_{(2\lambda)} - 4G_{(\lambda)}] < \infty \tag{9}$$

Relations (7-9) can be easily understood observing that the quantum potential owns a membrane elastic-like behavior where a higher curvature of the WFMS leads to higher energy. The fact that white fluctuations of the WFMS brings to a zero curvature wrinkles of the WFMS (and hence an infinite quantum potential energy) does not allow the realization of such a condition. The fact that independent fluctuations on smaller and smaller distance are progressively suppressed, means that exists a characteristic distance (let's name λ_c) above which the WFMS fluctuations are restrained in order to maintain the system energy finite.

Developing $G_{(\lambda)}$ in series expansion (for small Θ) as a function of λ/λ_c, where λ_c is analytically defined further on, such as

$$\lim_{\lambda \to 0} G_{(\lambda)} \cong a_0 + a_1\frac{\lambda}{\lambda_c} + a_2(\frac{\lambda}{\lambda_c})^2 + a_3(\frac{\lambda}{\lambda_c})^3 + a_4(\frac{\lambda}{\lambda_c})^4 + \sum_{j=5}^{\infty} a_j(\frac{\lambda}{\lambda_c})^j \tag{10}$$

it follows that (7-9) are verified if $a_0 = 1$, $a_1 = 0$, and $a_3 = 0$, while no condition applies to the coefficients a_2 and a_n with $n \geq 4$ that are unable to produce the divergence of (7-9) and remain undefined. Therefore, $G_{(\lambda)}$ reads

$$\lim_{\lambda \to 0} G_{(\lambda)} \cong 1 + a_2 \left(\frac{\lambda}{\lambda_c}\right)^2 + a_4 \left(\frac{\lambda}{\lambda_c}\right)^4 + \sum_{j=5}^{\infty} a_j \left(\frac{\lambda}{\lambda_c}\right)^j . \qquad (11)$$

where without a leaking of generality we can put $a_2 = \pm 1$ by a re-definition of the spatial cell side λ such as $\lambda' = a_2^{1/2} \lambda$.

In order to obtain a model holding also for a large-scale approach, we investigate in detail the model with $a_2 = -1$ ($a_2 = 1$ does not warrant the ergodicity) with the shape of the correlation function that reads

$$G_{(\lambda)} = exp[-(\frac{\lambda}{\lambda_c})^2] . \qquad (12)$$

Thence, from (5) we obtain

$$< \eta_{(q_\alpha, t)}, \eta_{(q_\beta + \lambda, t+\tau)} > = \underline{\mu} \frac{8m(k\Theta)^2}{\pi^3 \hbar^2} exp[-(\frac{\lambda}{\lambda_c})^2] \delta(\tau) \delta_{\alpha\beta} \qquad (13)$$

where [25]

$$\lambda_c = (\frac{\pi}{2})^{3/2} \frac{\hbar}{(2mk\Theta)^{1/2}} \qquad (14)$$

and where $\underline{\mu}$ is the WFMS mobility form factor that depends by specificity of the considered system [25].

2.1 Quantum non-locality length λ_L

In addition to the noise correlation function (12), to obtain the large-scale form of equations (15-23) we need to investigate the importance of the quantum force $\dot{p}_{qu} = -\nabla_q V_{qu}$ in (2) in the large-distance limit.

As shown in reference [25] the relevance of the force generated by the quantum potential (i.e., $| \nabla_q V_{qu} |$) at large distance can be evaluated by the convergence of the integral

$$\int_0^\infty | q^{-1} \nabla_q V_{qu} | dq \qquad (15)$$

If the quantum potential force grows less than a constant at large distance so that $\lim_{|q| \to \infty} | q^{-1} \nabla_q V_{qu} | \propto | q |^{-(1+\varepsilon)}$, where $\varepsilon > 0$, the integral (15) converges. In this case, the quantum potential range of interaction can be evaluated by the mean weighted distance

$$\lambda_L = 2 \frac{\int_0^\infty | q^{-1} \frac{dV_{qu}}{dq} | dq}{\lambda_c^{-1} | \frac{dV_{qu}}{dq} |_{(q=\lambda_c)}} . \qquad (16)$$

It is worth noting that for Gaussian-type states (as for linear systems) owing a quadratic quantum potential, so that it holds

$$\lim_{|q|\to\infty} |q^{-1}\nabla_q V_{qu}| = Constant$$

λ_L results infinite. In this case, the quantum non-local behavior is also effective on the large scale dynamics.

2.2 Schrödinger equation from the SQHA

For $\Theta = 0$ equation (1-3) with the identities

$$\dot{q} = \frac{\nabla_q S}{m} \tag{17}$$

where

$$S = \int_{t_0}^{t} dt\left(\frac{p \bullet p}{2m} - V_{(q)} - V_{qu}\right) \tag{18}$$

and

$$\mathrm{n}_{(q,t)} = A^2{}_{(q,t)} \tag{19}$$

can be derived [26] by the system of two coupled differential equations that read

$$\partial_t \mathrm{S}_{(q,t)} = -V_{(q)} + \frac{\hbar^2}{2m}\frac{\nabla_q{}^2 \mathrm{A}_{(q,t)}}{\mathrm{A}_{(q,t)}} - \frac{1}{2m}(\nabla_q \mathrm{S}_{(q,t)})^2 \tag{20}$$

$$\partial_t \mathrm{A}_{(q,t)} = -\frac{1}{m}\nabla_q \mathrm{A}_{(q,t)} \bullet \nabla_q \mathrm{S}_{(q,t)} - \frac{1}{2m} A \nabla_q{}^2 \mathrm{S}_{(q,t)} \tag{21}$$

that for the complex variable

$$\psi_{(q,t)} = \mathrm{A}_{(q,t)} \exp\left[\frac{\mathrm{i}}{\hbar}\mathrm{S}_{(q,t)}\right] \tag{22}$$

is equivalent to set to zero the real and imaginary part of the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_q{}^2\psi + V_{(q)}\psi \tag{23}$$

For $\Theta \neq 0$ the stochastic equations (1-3) can be obtained by the following system of differential equations

$$\partial_t \mathrm{S}_{(q,t)} = -V_{(q)} + \frac{\hbar^2}{2m}\frac{\nabla_q{}^2 \mathrm{A}_{(q,t)}}{\mathrm{A}_{(q,t)}} - \frac{1}{2m}(\nabla_q \mathrm{S}_{(q,t)})^2 \quad (24)$$

$$\partial_t \mathrm{A}_{(q,t)} = -\frac{1}{m}\nabla_q \mathrm{A}_{(q,t)} \bullet \nabla_q \mathrm{S}_{(q,t)} - \frac{1}{2m} A \nabla_q{}^2 \mathrm{S}_{(q,t)} + A^{-1}\eta_{(q_\alpha,t,\Theta)} \quad (25)$$

that for the complex variable (22) are equivalent to the stochastic Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_q{}^2\psi + V_{(q)}\psi + i\frac{\psi}{|\psi|^2}\eta_{(q_\alpha,t,\Theta)}. \quad (26)$$

2.3 Limiting dynamics

Generally speaking, given a problem with a physical length ΔL, depending by the two lengths λ_c and λ_L , built-in into the SQHA, various limiting dynamics follow:

1) *Non-local deterministic dynamics* (i.e., the standard quantum mechanics) with ΔL << $\lambda_c \cup \lambda_L$ (e.g., $\Theta \to 0$). In this case it follows that

$$\partial_t \mathrm{n}_{0(q,t)} = -\nabla_q \bullet (\mathrm{n}_{0(q,t)}\dot{q}) \quad (27)$$

and that

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_q{}^2\psi + V_{(q)}\psi$$

2) *Non-local stochastic dynamics*, with λ_c << ΔL << λ_L

$$\partial_t \mathrm{n}_{(q,t)} = -\nabla_q \bullet (\mathrm{n}_{(q,t)}\nabla_q \dot{q}) + \eta_{(q_\alpha,t,\Theta)} \quad (28)$$

$$< \eta_{(q_\alpha,t)}, \eta_{(q_\alpha+\lambda,t+\tau)} > = \underline{\mu}\,\delta_{\alpha\beta}\frac{2k\Theta}{\lambda_c}\delta(\lambda)\delta(\tau) \quad (29)$$

In this case the stochastic Schrödinger equation (26) reads

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_q{}^2\psi + V_{(q)}\psi + i\frac{\psi}{|\psi|^2}\eta_{(q_\alpha,t,\Theta)} \quad (30)$$

$$< \eta_{(q_\alpha,t)}, \eta_{(q_\alpha+\lambda,t+\tau)} > = \underline{\mu}\,\delta_{\alpha\beta}\frac{2k\Theta}{\lambda_c}\delta(\lambda)\delta(\tau) \quad (31)$$

3) *Local stochastic dynamics*, with $\lambda_c \cup \lambda_L$ <<ΔL.
Given the condition λ_L <<ΔL so that it holds

$$\lim_{|q|\to\infty} |\nabla_q V_{qu(\mathrm{n}_0)}| = 0 \qquad (32)$$

the SPDE of motion acquires the form

$$\partial_t \mathrm{n}_{(q,t)} = -\nabla_q \bullet (\mathrm{n}_{(q,t)} \dot{q}) + \eta_{(q_\alpha,t,\Theta)} \qquad (33)$$

$$< \eta_{(q_\alpha,t)}, \eta_{(q_\alpha+\lambda,t+\tau)} >= \underline{\mu}\, \delta_{\alpha\beta} \frac{2k\Theta}{\lambda_c} \delta(\lambda)\delta(\tau) \qquad (34)$$

$$\dot{q} = \frac{p}{m} = \nabla_q \lim_{\Delta\Omega/\lambda_L \to\infty} \frac{\nabla_q S}{m} = \nabla_q \{ \lim_{\Delta\Omega/\lambda_L \to 0} \frac{1}{m} \int_{t_0}^{t} dt(\frac{p \bullet p}{2m} - V_{(q)} - V_{qu(\mathrm{n})} - I^*) \}$$
$$= \frac{1}{m} \nabla_q \{ \int_{t_0}^{t} dt\, (\frac{p \bullet p}{2m} - V_{(q)} - \Delta) \} = \frac{p_{cl}}{m} + \frac{\delta p}{m} \cong \frac{p_{cl}}{m} \qquad (35)$$

where δp is a small fluctuation of the momentum and

$$\dot{p}_{cl} = -\nabla_q V_{(q)}. \qquad (36)$$

In this case, by using the identities (17-19) we can write

$$\partial_t \mathrm{S} = -V_{(q)} - \frac{1}{2m} (\nabla_q \mathrm{S})^2 \qquad (37)$$

$$\partial_t \mathrm{A}_{(q,t)} = -\frac{1}{m} \nabla_q \mathrm{A}_{(q,t)} \bullet \nabla_q \mathrm{S}_{(q,t)} - \frac{1}{2m} A \nabla_q{}^2 \mathrm{S}_{(q,t)} + A^{-1} \eta_{(q_\alpha,t,\Theta)} \qquad (38)$$

Clearly, it is not possible to obtain the Schrödinger equation by (37-38) since S given by (35) converges to the classical value S_{cl}

$$S_{cl} = \int_{t_0}^{t} dt\, (\frac{p \bullet p}{2m} - V_{(q)}). \qquad (39)$$

Nevertheless, for the wave function $\psi_{(q,t)}$ the classical stochastic equation of motion (37-38) can be cast in a non-linear Schrödinger equation that reads:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} (\nabla_q{}^2 \psi - \frac{\psi}{|\psi|} \nabla_q{}^2 |\psi|) + V_{(q)} \psi + i \frac{\psi}{|\psi|^2} \eta_{(q_\alpha,t,\Theta)}. \qquad (40)$$

The former differential equation describes the evolution of a particle spatial density $|\psi|$ owing a classical action S_{cl}. Actually, the exact equation is given by (26) while the former one (40) is just a limiting one and the formal transformation between them is just intrinsic.

Alternatively, in order to describe phenomena at the edge between the classical and the quantum behavior, a semi-empirical equation for passing from (26) to (40) could be more manageable.

By considering that the when the physical length of the system ΔL is much smaller than the quantum non-locality length λ_L, the system is quantum, while when λ_L is very small compared to ΔL is classic, it is possible to write

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}(\nabla_q{}^2\psi-\alpha\frac{\psi}{|\psi|}\nabla_q{}^2|\psi|)+V_{(q)}\psi+i\frac{\psi}{|\psi|^2}\eta_{(q_\alpha,t,\Theta)} \qquad (41)$$

where α at first order in a series expansion as a function of the dimensionless parameter $\frac{\lambda_L}{\Delta L}$ reads

$$\alpha\cong\frac{\frac{\Delta L}{\lambda_L}}{1+\frac{\Delta L}{\lambda_L}} \qquad (42)$$

In the case when (41) is used to describe a system at the edge between the quantum and classical dynamics (i.e., $\frac{\lambda_L}{\Delta L}\approx 1$) we have $\alpha\cong\frac{1}{2}$.

It is interesting to note that Equation (41) for pseudo-Gaussian states that have the large-distance hyperbolic behavior

$$\lim_{|q|\to\infty}|\psi|\propto l^{-1/2}q^{-1} \qquad (44)$$

that holds for Lennard-Jones type potentials

$$V_{L-J(q)}=4V_0[(\frac{\sigma}{r})^{12}-(\frac{\sigma}{r})^6] \qquad (45)$$

such as in the ^{4}He dimer [27], equation (41) acquires the stochastic form of the Gross-Pitaevskii one [28]

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}(\nabla_q{}^2\psi-\frac{l}{4}|\psi|^2\psi)+V_{(q)}\psi+i\frac{\psi}{|\psi|^2}\eta_{(q_\alpha,t,\Theta)} \qquad (46)$$

3. DISCUSSION

The existence of λ_L finite allows fluctuations, as small as we like, to overcome the quantum force on large distance so that the quantum non-locality can only be maintained on a finite distance of order of λ_L . Since λ_L finite can steam out from a large number of real non-linear potentials, while the case of an infinite quantum non-locality length (such as in the linear case) actually seems to be an exception, the classical mechanics realizes itself on the scale of macroscopic physics. Generally speaking, it must be observed that even thought fluctuations are present, in the case of an infinite λ_L, they are not sufficient to break the quantum mechanics and to lead to the classical one. Under this light, the macro-scale description is not sufficient to obtain the classical behavior if not coupled to a finite quantum non-locality. With this respect, the WKB approximation is an illuminating example being the non-local large-scale description but the classical limit.

Fluctuations may break quantum non-locality in non-linear systems (λ_L finite) when, in this case, the quantum pseudo-potential decreases with distance and, beyond the non-locality length λ_L, it becomes much smaller than its fluctuations and can be neglected. It must be mentioned that, only in the stochastic approach the quantum potential can be correctly neglected while it cannot be taken off by the deterministic equation (1-3) because in such a case this operation will change the structure of the equation [25] destroying the quantum stationary states (i.e., eigenstates) and deeply changing the evolution of the system in a sufficiently short interval of time.

4. CONCLUSION

The SQHA shows that the quantum potential in presence of spatial noise modifies the shape of the fluctuations of the WFMS suppressing them on a distance much shorter than a characteristic length (of quantum coherence) $\lambda_c = (\frac{\pi}{2})^{3/2} \frac{\hbar}{(2mk\Theta)^{1/2}}$. As a consequence of that, the quantum mechanics is achieved when the physical scale of the problem is much smaller than λ_c (e.g., the deterministic limit of null noise amplitude $\Theta = 0$). The correlation function of the WFMS fluctuations (and its characteristic distance λ_c) for small noise amplitude Θ , has been derived by imposing that the system energy in the fluctuating state does not diverge but remains finite. The model highlights that in the stochastic case, beyond the quantum coherence length λ_c, the non-local quantum potential may have a finite range of interaction maintaining the quantum non-local behavior on a distance of order of "quantum non-locality length" λ_L . The value of λ_L depends both by the fluctuations amplitude and by the inter-particle law of interaction: systems interacting by linear or stronger forces have an infinite range of quantum potential interaction; system with weaker a potential, such as the Lennard-Jones one can have a finite λ_L . For non-linear interactions, the noise may produce quantum non-locality breaking when the force of the quantum potential decreases and becomes vanishing at large distance (beyond λ_L finite) becoming negligible with respect to the fluctuations. For $\hbar \neq 0$ the classical stochastic behavior is achieved when λ_c as well as λ_L are negligibly small with respect to the physical length of the problem, while for the (unphysical) case of $\hbar = 0$ the deterministic classical mechanics is realized. The SQHA model furnishes a unified approach for the quantum and the classical behavior. The quantum mechanics is deterministic (at glance with satisfying philosophical requirements of the quantum mechanics) while the classical one is achieved when, beyond λ_L, fluctuations overcome and disrupt the quantum potential force in building up the quantum eigenstates.

Finally, the SQHA is able to give a theoretical support to the formulation of the semi-empirical Gross-Pitaevskii equation needed to describe the open quantum mechanics where the environmental fluctuations and the classical effects start to be relevant.

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NOMENCLATURE

n : squared wave function modulus	l^{-3}
S : action of the system	$m^{-1}\ l^{-2}\ t$
m : mass of structureless particles	m
$\hbar$: Plank’s constant	$m\ l^{2}\ t^{-1}$
c : light speed	$l\ t^{-1}$
k : Boltzmann’s constant	$m\ l^{2}\ t^{-2}$/°K
Θ : Noise amplitude	°K
H : Hamiltonian of the system	$m\ l^{2}\ t^{-2}$
V : potential energy	$m\ l^{2}\ t^{-2}$
V_{qu} : quantum potential energy	$m\ l^{2}\ t^{-2}$
η : Gaussian noise of particle density	$l^{-3}\ t^{-1}$
λ_c : correlation length of squared wave function modulus fluctuations	l
λ_L : range of interaction of non-local quantum interaction	l
G(λ) : dimensionless correlation function (shape) of WFMS fluctuations	pure number
$\underline{\mu}$: WFMS mobility form factor	$m^{-1}\ t\ \ l^{-6}$
μ = WFMS mobility constant	$m^{-1}\ t$