# Quantum Gravity and the Holographic Mass

**Nassim Haramein**[†]

*Hawaii Institute for Unified Physics*

## ABSTRACT

We find an exact quantized expression of the Schwarzschild solution to Einstein's field equations utilizing spherical Planck units in a generalized holographic approach. We consider vacuum fluctuations within volumes as well as on horizon surfaces, generating a discrete spacetime quantization and a novel quantized approach to gravitation. When applied at the quantum scale, utilizing the charge radius of the proton, we find values for the rest mass of the proton within $0.069 \times 10^{-24} \, gm$ of the CODATA value and when the recent muonic proton charge radius measurement is utilized we find a deviation of $0.001 \times 10^{-24} \, gm$ from the proton rest mass. We identify a fundamental mass ratio between the vacuum oscillations on the surface horizon and the oscillations within the volume of a proton and find a solution for the gravitational coupling constant to the strong interaction. We derive the energy, angular frequency, and period for such a system and determine its gravitational potential considering mass dilation. We find the force range to be closely correlated with the Yukawa potential typically utilized to illustrate the exponential drop-off of the confining force. Zero free parameters or hidden variables are utilized.

*Keywords: Quantum gravity, holographic principle, Schwarzschild solution, proton charge radius, strong interaction, Yukawa potential*

## 1. INTRODUCTION

In 1916, Karl Schwarzschild published an exact solution to Einstein's field equations for the gravitational field outside a spherically symmetric body [1,2]. The Schwarzschild solution determined a critical radius, $r_s$ for any given mass where the escape velocity equals $c$, the speed of light. The region where $r = r_s$ is typically denoted as the horizon or event horizon and is given by the well known definition

$$r_s = \frac{2Gm}{c^2} \tag{1}$$

where $G$ is the gravitational constant, and $m$ is the mass. John Archibald Wheeler in 1967 described this region of space as a "*black hole*" during a talk at the NASA Goddard Institute of Space Studies. In 1957 Wheeler had already, as an implication of general relativity, theorized the presence of tunnels in spacetime or "*wormholes*" and in 1955, as a consequence of quantum mechanics, the concept of "*spacetime foam*" or "*quantum foam*" as a qualitative description of

28 subatomic spacetime turbulence [3].  The theory predicts that the very fabric of spacetime is a
29 seething foam of wormholes and tiny virtual black holes at the Planck scale as well as being the
30 source of virtual particle production.  In Wheeler's own words: *"The vision of quantum gravity is a*
31 *vision of turbulence – turbulent space, turbulent time, turbulent spacetime… spacetime in small*
32 *enough regions should not be merely "bumpy," not merely erratic in its curvature; it should*
33 *fractionate into ever-changing, multiply-connected geometries.  For the very small and the very*
34 *quick, wormholes should be as much a part of the landscape as those dancing virtual particles*
35 *that give to the electron its slightly altered energy and magnetism* [Observed as the Lamb shift]*."*
36 [4]

38 On the cosmological scale, black hole singularities were initially thought to have no physical
39 meaning and probably did not occur in nature.  As general relativity developed in the late 20th
40 century it was found that such singularities were a generic feature of the theory and evidence for
41 astrophysical black holes grew such that they are now accepted as having physical existence and
42 are an intrinsic component of modern cosmology.  While the Schwarzschild solution to Einstein's
43 field equations results in extreme curvature at the origin and the horizon of a black hole, it is
44 widely utilized to give appropriate results for many typical applications from cosmology to
45 planetary physics.  For instance, the Newtonian gravitational acceleration near a large, slowly
46 rotating, nearly spherical body can be derived by $g = r_s c^2 / 2r^2$  where $g$ is the gravitational
47 acceleration at radial coordinate $r$, $r_s$ is the Schwarzschild radius of a gravitational central body,
48 and $c$ is the speed of light.  Similarly, Keplerian orbital velocity can be derived for the circular
49 case by

$$v = \sqrt{\frac{r_s c^2}{2r}}$$
(2)

51 where $r$ is the orbital radius.  This can be generalized to elliptical orbits and of course the
52 Schwarzschild radius is utilized to describe relativistic circular orbits or photon spheres for rapidly
53 rotating objects such as black holes.  There are many more examples of the ubiquitous nature of
54 the Schwarzschild solution and its applications to celestial mechanics and cosmology.

56 In developments over the past decade event horizons have been demonstrated to be dynamically
57 fluctuating regions at a scale where quantum mechanical effects occupy a central role.  Early
58 explorations of spacetime fluctuations at the quantum level predicted that the vacuum at those
59 scales undergoes extreme oscillations as formulated in the Wheeler model.  Indeed, in quantum
60 field theory, the vacuum energy density is calculated by considering that all the vibrational modes
61 have energies of $\hbar\omega/2$.  When summed over all field modes, an infinite value results unless
62 renormalized utilizing a Planck unit cutoff [5].  Yet, while the high curvature of general relativity
63 and the vacuum fluctuations of quantum field theory converge and meet at the Planck cutoff,
64 efforts to define gravitational curvature in a discrete and elegant manner, as in quantum gravity
65 have proven elusive.

67 In the early 1970s, expanding from Hawking temperature theorems for black hole horizons,
68 Bekenstein conjectured that the entropy of a black hole is proportional to the area of its event
69 horizon divided by the Planck area times a constant on the order of unity [6].  Hawking confirmed
70 Bekenstein's conjecture utilizing the thermodynamic relationships between energy and
71 temperature [7].

$$S = \frac{kA}{4\ell^2}$$
(3)

73 where $A$ is the area of the event horizon, $k$ is Boltzmann's constant, and $\ell$ is the Planck length.
74 The Bekenstein bound conjecture and the entropy of a black hole eventually led to the
75 holographic principle (generally given as an analogy to a hologram by Gerard 't Hooft) [8] where
76 the covariant entropy bound demands that the physics in a certain region of space is described
77 by the information on the boundary surface area, where one bit is encoded by one Planck area

78    [8,9]. Since the temperature $T_H = \dfrac{k}{2\pi}$ determines the multiplicative constant of the Bekenstein-

79    Hawking entropy of a black hole which is

80
$$S = \frac{A}{4} \tag{4}$$

81    therefore, Hawking fixes the proportionality constant at $1/4$ of the surface area, which we note is
82    equivalent to the surface area of the equatorial disc of the system.
83

84    In this paper, we generalize the holographic principle by utilizing a spherical Planck unit rather
85    than a surface area Planck unit, $\ell^2$ as a minimum-size vacuum energy oscillator on which
86    information encodes, which we term "Planck spherical unit" (PSU). This approach is consistent
87    with the dimensional reduction of the holographic principle, which states explicitly that all the
88    information of the interior volume of a black hole is encoded holographically on it's horizon
89    surface. We consider the interior vacuum energy density ratio, in terms of PSU packing, to the
90    surface horizon and find a generalized holographic principle which broadens the applicability of
91    the holographic method to other areas of physics, such as gravitation, hadronic mass, and
92    confinement.
93

94    As a result, an exact quantized derivation of the Schwarzschild solution to Einstein's field
95    equations is found, yielding a novel approach to quantum gravity. We apply this method to the
96    quantum scale and derive the proton rest mass from geometric considerations alone. When the
97    CODATA charge radius value of the proton is employed, our result yields a very close first-order
98    approximation within ~4% deviation from the CODATA mass value, the difference of which is
99    $0.069 \times 10^{-24} \, gm$. Utilizing the recent muonic measurement of the proton charge radius however
100    [10], we obtain a more accurate value within $0.001 \times 10^{-24} \, gm$ or ~0.07% deviation. Employing
101    our generalized holographic approach we predict a precise proton charge radius. Our prediction
102    falls within the reported experimental uncertainty for the muonic measurement of the proton
103    charge radius [10].
104

105    By further algebraic derivation, we find a fundamental constant we term $\phi$, defined by the mass
106    ratio of vacuum oscillations on the surface horizon to the ones within the volume of the proton.
107    As a result, clear relationships emerge between the Planck mass, the rest mass of the proton,
108    and the Schwarzschild mass of the proton or what we term the holographic gravitational mass.
109    Further, we find that our derived fundamental constant $4\phi^2$ generates the gravitational coupling
110    constant to the strong interaction, thus defining the unification energy for confinement. We also
111    derive the energy, angular frequency, and period for such a system utilizing our generalized
112    holographic approach. We find that the period is on the order of the interaction time of particle
113    decay via the strong force which is congruent with our derivation of the gravitational coupling
114    constant. Moreover, the frequency of the system correlates well with the characteristic gamma
115    frequency of the nucleon decay rate. Finally, we compute the gravitational potential resulting
116    from the mass dilation of the system due to angular velocities as a function of radius and find that
117    the gravitational force of such a system produces a force range drop-off closely correlated with
118    the Yukawa potential typically utilized to define the short range of the strong interaction.
119

120    We demonstrate that a quantum gravitational framework of a discrete spacetime defined by
121    spherical Planck vacuum oscillators can be constructed which applies to both cosmological and
122    quantum scales. Our generalized holographic method utilizes zero free parameters and is
123    generated from simple geometric relationships and algebra, yielding precise results for significant
124    physical properties such as the mass of black holes, the rest mass of the proton, and the
125    confining nuclear force.
126

127 Note that in this paper, we utilize the full significant digits of the Planck length and other relevant
128 physical constants as given by CODATA in our derivations to demonstrate the accuracy of our
129 results.
130
131 **2. THE SCHWARZSCHILD SOLUTION FROM PLANCK OSCILLATOR SPHERICAL**
132 **UNITS**
133
134 In view of the increasingly significant role that quantum field effects or vacuum fluctuations have
135 played in current cosmology to characterize the information structure of the horizons of
136 astrophysical black holes, as in the holographic principle and its application to entropy [11], we
137 examine a hypothetical black hole horizon of the approximate order of magnitude of the well
138 documented black hole Cygnus X-1 with a radius of $\sim 2.5 \times 10^6 cm$.
139
140 In order to better represent the natural systems of harmonic oscillators we initiate our calculation
141 by defining a Planck spherical unit (PSU) oscillator of the Planck mass $m_\ell$ with a spherical
142 volume $V_{\ell s}$ and a Planck length diameter $\ell = 1.616199 \times 10^{-33} cm$ with a radius of $\ell_r = \ell / 2$.
143 We utilize a spherical volume for our fundamental spacetime quantum foam PSU oscillator
144 instead of the typical Planck area $\ell^2$ or Planck volume $\ell^3$ in our generalized holographic
145 approach. Therefore a spherical PSU of radius $\ell_r$ has a volume of

$$V_{\ell s} = \frac{4}{3} \pi \ell_r^3 \qquad (5)$$

147 or $V_{\ell s} = 2.210462 \times 10^{-99} cm^3$. Such a sphere will have an equatorial plane circular area of

$$A_{\ell c} = \pi \ell_r^2 \qquad (6)$$

149 or $A_{\ell c} = 2.051538 \times 10^{-66} cm^2$, which will be utilized for the purpose of holographic tiling. In our
150 generalized holographic approach we consider the volume vacuum oscillation energy in terms of
151 Planck spherical units as well as the typical tiling of the surface horizon found in the holographic
152 principle entropy calculations of equations (3) and (4). Our considerations of information within
153 the volume stems from an exploration of the role of vacuum fluctuations in surface gravity and
154 spacetime quantization relationships between the interior information network and the external
155 surface tiling. It is important to note that although, in this exercise, we tile the surface horizon
156 with Planck circular areas, these are equatorial areas of spherical oscillators.
157
158 Consequently, we derive the quantity $\eta$, the number of Planck areas $A_{\ell c}$ on the surface $A$ of
159 the horizon of Cygnus X-1 with a radius of $2.5 \times 10^6 cm$ and find that

$$\eta = \frac{A}{A_{\ell c}} \qquad (7)$$

161 or $\eta = 3.828339 \times 10^{79}$. We calculate $R$ or the quantity of Planck volume oscillators $V_{\ell s}$ within
162 the volume $V$ of the interior of the Cygnus X-1 black hole

$$R = \frac{V}{V_{\ell s}} \qquad (8)$$

164 or $R = 2.960912 \times 10^{118}$. We then examine the relationship between the information network of
165 the horizon $\eta$ and the interior information network of PSU oscillators $R$, then multiply it by the
166 Planck mass, $m_\ell$ to obtain the mass-energy equivalence of the ratio and we determine that

$$m_h = \frac{R}{\eta} m_\ell \qquad (9)$$

4

168  where $m_h = 1.683354 \times 10^{34} gm$ is the mass derived from this geometric approach, or what we
169  term the "*holographic gravitational mass*". This expression can be written as well in terms of
170  mass relations by multiplying equation (9) by $m_\ell / m_\ell$

$$m_h = \frac{R_\rho}{\eta_\rho} m_\ell \qquad (10)$$

172  where $R_\rho$ is the total mass-energy of PSU oscillators within the volume and $\eta_\rho$ is the mass-
173  energy of PSU oscillators on the surface horizon, so that all terms are Planck mass quantities,
174  which clarifies the relationship between masses in the geometry. Equation (10) can then be
175  written as

$$m_h = \frac{R_\rho}{\eta} . \qquad (11)$$

177  We then calculate the Schwarzschild mass of a black hole of the same radius as our example
178  Cygnus X-1. Rearranging equation (1) we have

$$\frac{rc^2}{2G} = m_s \qquad (12)$$

180  where $m_s$ is the Schwarzschild mass of such a black hole, $c$ is the speed of light and $G$ is the

181  gravitational constant. We obtain the exact same quantity, $m_s = 1.683354 \times 10^{34} gm$ utilizing
182  CODATA values.
183  Therefore

$$m_h = m_s . \qquad (13)$$

185  We find that a simple relationship of the internal PSUs within a given volume, to the discrete
186  "pixelation" of the holographic membrane surface horizon of the black hole yields what we term
187  the *holographic gravitational mass* of the object which is equivalent to its classical Schwarzschild
188  mass. This of course, is valid for any system, is free of any relativistic expressions, and utilizes
189  only discrete Planck quantities, which has implications for quantum gravity.
190
191  From the above geometric analysis we then perform an algebraic derivation to find an elegant
192  formulation of this quantized relationship. Therefore we can write equation (11) in terms of
193  equation (7) and $R$

$$\frac{R_\rho}{\eta} = \frac{Rm_\ell}{A / A_{\ell c}} = \frac{Rm_\ell A_{\ell c}}{A} . \qquad (14)$$

195  Utilizing equations (6) and (8) and rearranging terms we have

$$= \frac{(V / V_{\ell s}) m_\ell \pi \ell_r^2}{4 \pi r^2} = \frac{(V / V_{\ell s}) m_\ell \ell_r^2}{4 r^2} . \qquad (15)$$

197  Expanding to the spherical form in terms of $r$ and $\ell_r$ and reducing,

$$= \frac{(\tfrac{4}{3} \pi r^3 / (\tfrac{4}{3} \pi \ell_r^3)) m_\ell \ell_r^2}{4 r^2} = \frac{(r^3 / \ell_r^3) m_\ell \ell_r^2}{4 r^2} \qquad (16)$$

199  or,

$$\frac{R_\rho}{\eta} = r \frac{m_\ell}{4 \ell_r} \qquad (17)$$

201  where $r$ is the radius of a system. Given that $\ell_r = \ell / 2$, and utilizing equation (11) we now

202  obtain what we have previously termed the holographic gravitational mass $m_h$ as,

203
$$r \frac{m_\ell}{2\ell} = m_h .$$
(18)

204 Of course now a radius we term the *holographic radius* $r_h$ can be calculated for any mass $m$,
205 giving the expression

206
$$r_h = 2\ell \frac{m}{m_\ell} .$$
(19)

207 Therefore, we find that the number of discrete Planck masses within any given mass $m$
208 multiplied by $2\ell$, which is a discrete quantity, will generate the *holographic radius* equivalent to
209 the well known Schwarzschild radius of equation (1) so that in the case of equation (19) we have
210 a non-relativistic form derived from discrete vacuum oscillator Planck quantities generating a
211 quantized solution. The geometric equation (9) and the algebraic derivation (19) are both simple
212 and meaningful as they clearly demonstrate that the gravitational mass of an object can be
213 obtained from discrete quantities based on Planck spherical units. Consequently our results are
214 consistent with the dimensional reduction embodied in the holographic principle, and thus we
215 have found a unique expression involving the holographic gravitational mass, radius, Planck
216 mass, and the mass of any black-hole object that is congruent with the usual holographic entropy
217 computation of equation (3) and (4).
218

219 Clearly in both cases $c$ and $G$ are involved since Planck entities are derived from $\ell = \sqrt{\dfrac{\hbar G}{c^3}}$

220 and $m_\ell = \sqrt{\dfrac{\hbar c}{G}}$, therefore we can write equation (19) as

221
$$r_h = 2m \frac{\ell}{m_\ell} = 2m \frac{\sqrt{\dfrac{\hbar G}{c^3}}}{\sqrt{\dfrac{\hbar c}{G}}} = 2m \sqrt{\dfrac{G^2}{c^4}}$$
(20)

222 or

223
$$r_s = r_h = \frac{2Gm}{c^2} .$$
(21)

224 Here we arrive to the Schwarzschild expression of equation (1) from geometric considerations
225 alone. It then follows that the Schwarzschild solution to Einstein's field equations could have
226 been developed in the late 19th Century by computation of tiling Planck quantities independent of
227 spacetime curvature and singularities, near the time when Max Planck in 1899 derived his units.
228 His units were, of course, the result of the renormalization of the electromagnetic spectrum of
229 black body radiation by the utilization of a quantum of action $h$, which confirmed experimental
230 results. Planck quantities are natural units, free of any arbitrary anthropocentric measurements,
231 are based on fundamental physical constants, and can be defined as, for example, the time it
232 takes a photon to travel one Planck length which is the Planck time. Therefore, in the case of the
233 generalized holographic solution the difficulties associated with discontinuities and singularity
234 production are precluded from occurring due to the Planck quantization where the presence of $\hbar$,
235 the quantum of angular momentum or the quantum of action of the energetic vacuum quantizes
236 spacetime and yields a discrete gravitational mass or quantum gravity.
237

238 However, if our holographic solution is a correct representation of quantum gravitational
239 spacetime structure, then it should be applicable to the quantum world and yield appropriate
240 results such as fundamental physical quantities from first principles and geometric considerations.
241

## 3.  HOLOGRAPHIC MASS AT THE HADRON SCALE

We now apply the above surface to volume relationships of Planck vacuum oscillations of a cosmological scale object to the quantum world.  We initially utilize the standard CODATA proton charge radius given as $r_p = 0.8775 \times 10^{-13} cm$ due to the fundamental nature of protons in the hadronic picture.  We derive the quantity $\eta$ as the number of Planck areas $A_{\ell c}$ on the surface area $A_p$ of a proton

$$\eta = \frac{A_p}{A_{\ell c}}. \tag{22}$$

In this case, $\eta = 4.716551 \times 10^{40}$.  Multiplying by the Planck mass, we obtain

$$\eta_\rho = \eta m_\ell = 1.026562 \times 10^{36} gm \tag{23}$$

or the holographic mass of the surface horizon of the proton.  We then calculate $R$ or the number of PSUs within the proton volume $V_p$ utilizing equation (8), yielding $R = 1.280404 \times 10^{60}$.

We can now examine the relationship between $\eta_\rho$ and $R$ and find

$$m_{p'} = 2 \frac{\eta_\rho}{R} = 1.603498 \times 10^{-24} gm \tag{24}$$

where $m_{p'}$ is the holographic derivation of the mass of the proton.  The result is a close approximation to the measured CODATA value for the proton mass $m_p = 1.672622 \times 10^{-24} gm$ with a $0.069 \times 10^{-24} gm$ or ~4% deviation from the CODATA value.

Therefore a simple reversal of the holographic "pixelation" relationship in equation (11) produces a close approximation to the rest mass of the proton; whereas the above geometric holographic gravitational mass (which is equivalent to the Schwarzschild solution) is generated by dividing the mass of PSUs in the interior by the number of PSUs on the surface, conversely the proton rest mass is extrapolated from the mass of PSUs on the surface divided by the number of PSUs in the interior.  Clearly both equation (11) and it's inverse in equation (24) can be utilized to describe a relationship between the interior information to the screening on the surface horizon and is consistent with the dimensional reduction associated with the holographic approach.  In the following sections we will clarify the nature of this relationship, which has significant implications to the gravitational coupling constant and confinement.

The usual method of determining the charge radius of the proton is to measure the Lamb shift of a bound proton-lepton system via spectroscopy.  A prior method was to measure the Sachs electric form factor with a scattering experiment, such as electron-proton scattering.  The Sachs form factors are the spatial Fourier transforms of the proton's charge distribution in the Breit frame [12].  Recently an international research team from the Paul Scherrer Institut (PSI) in Villigen (Switzerland) and scientists from the Max Planck Institute of Quantum Optics (MPQ) in Garching, the Ludwig-Maximilians-Universität (LMU) Munich and the Institut für Strahlwerkzeuge (IFWS) of the Universität Stuttgart (both from Germany), and the University of Coimbra, Portugal obtained measurements recently published in *Nature* of the spectrum of muonic hydrogen that found a significantly lower value of $r_p = 0.84184 \times 10^{-13} cm$ [10] compared to the CODATA value of the proton charge radius.  In the case of measuring the Lamb shift of a bound proton-muon system it was anticipated to reduce the error by an order of magnitude compared to measurements from proton-electron scattering and typical proton-electron spectroscopy [13].  While it did indeed reduce the error by an order of magnitude, the fact that the new measurement

286 is five standard deviations from the CODATA value has raised significant questions about the
287 implications of this new result on Quantum Electrodynamics, and so far no experimental errors
288 have been found despite thorough scrutiny by the physics community [14-20].

290 We now proceed to calculate the rest mass of the proton as above, utilizing the new muonic
291 hydrogen measured proton charge radius $r_p = 0.84184 \times 10^{-13} cm$ and find $\eta = 4.340996 \times 10^{40}$,
292 $\eta_\rho = 9.448222 \times 10^{35} gm$, and $R = 1.130561 \times 10^{60}$. Again utilizing equation (24) we obtain

$$m_{p'} = 2\frac{\eta_\rho}{R} = 1.6714213 \times 10^{-24} gm. \tag{25}$$

294 This result is now a much closer approximation to the measured CODATA value for the proton
295 mass $m_p = 1.672622 \times 10^{-24} gm$ with a $0.0012 \times 10^{-24} gm$ or ~0.07% deviation from the
296 CODATA value. This extremely close result is supportive of the new muonic hydrogen
297 measurement of the proton charge radius, and of our generalized holographic approach applied
298 to the quantum scale. Considering that this method yields an exact solution to the gravitational
299 mass of an object, we can now make a prediction of the precise radius of the proton from
300 theoretical tenets. Assuming that the current CODATA mass measurement of the proton (which
301 has been measured to a high level of precision empirically) is accurate, we can solve equation
302 (25) for the radius of an object of mass $m_p = 1.672622 \times 10^{-24} gm$ by utilizing algebraic
303 computations from the geometric consideration. Consequently

$$m_{p'} = 2\frac{\eta_\rho}{R} = 2\frac{(A/A_{\ell c})m_\ell}{V/V_{\ell s}}. \tag{26}$$

305 Substituting equations (5) and (6) on the right side and canceling common terms we have

$$= 2\frac{(4\pi r^2 / \pi \ell_r^2)m_\ell}{4/3 \pi r_p^3 / (4/3 \pi \ell_r^3)} = 2\frac{(4r^2 / \ell_r^2)m_\ell}{r_p^3 / \ell_r^3} \tag{27}$$

and reducing to

$$= \frac{8m_\ell}{r_p / \ell_r} = \frac{8\ell_r m_\ell}{r_p}. \tag{28}$$

309 Since $\ell_r = \ell/2$, we can reduce this to

$$m_{p'} = 4\ell\frac{m_\ell}{r_p}. \tag{29}$$

311 Therefore the mass of the proton can be simply extrapolated from the relationship of the Planck
312 length times the Planck mass divided by the proton charge radius. Again, as in section 2 we find
313 a simple and elegant quantized solution to a fundamental physical quantity utilizing an intrinsic
314 generalized holographic relationship.

316 We now can predict a precise radius for the proton, which we term $r_{p'}$, from the CODATA value
317 for the proton mass by inverting equation (29)

$$r_{p'} = 4\ell\frac{m_\ell}{m_p} = 0.841236 \times 10^{-13} cm \tag{30}$$

319 a difference of $0.000604 \times 10^{-13} cm$ from the muonic measurement of the proton charge radius of
320 $r = 0.84184(67) \times 10^{-13} cm$ and therefore falls within less than one standard deviation
321 $0.00067 \times 10^{-13} cm$, or within their reported standard experimental error value [10]. More precise
322 measurement may confirm this theoretical result.

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326 **4. DETERMINING A FUNDAMENTAL GEOMETRIC MASS RATIO AND THE**
327 **GRAVITATIONAL COUPLING CONSTANT**
328

329 As in section (2), we now replace $\ell$ and $m_\ell$ in equation (29) by their respective fundamental
330 constant Planck unit definitions, to derive deeper meaning.  Therefore, canceling terms and
331 simplifying

$$m_{p'} = 4\ell\frac{m_\ell}{r_p} = \frac{4\sqrt{\frac{\hbar G}{c^3}\frac{\hbar c}{G}}}{r_p} = \frac{4\sqrt{\frac{\hbar^2}{c^2}}}{r_p} = \frac{4\frac{\hbar}{c}}{r_p} = \frac{4\hbar}{r_p c}.$$ (31)

333 We rewrite the last term and multiply the numerator and denominator by $c/G$,

$$= 2\frac{\hbar}{r_p c/2} = 2\frac{\hbar c/G}{r_p c^2/2G}$$ (32)

335 and since $m_\ell = \sqrt{\frac{\hbar c}{G}}$, we substitute

$$= 2\frac{m_\ell^2}{r_p c^2/2G}.$$ (33)

337 Here the Schwarzschild condition $m_s = rc^2/2G$ appears in the denominator which is equivalent
338 to our holographic solution $m_h = rm_\ell/2\ell$. We can now write equation (33) as

$$m_{p'} = 2\frac{m_\ell^2}{m_{h'}}.$$ (34)

340 This is a significant result as we now observe a direct relationship between the rest mass of the
341 proton $m_p$, the Planck mass $m_\ell$, and the Schwarzschild mass or holographic gravitational mass
342 $m_h$, which we denote as $m_{h'}$ to indicate the holographic gravitational mass specific to the proton.
343 Thus, the presence of a strong gravitational potential equivalent to the Schwarzschild mass in
344 equation (34) relates the rest mass of the proton to our cosmological generalized holographic
345 mass solution, confirming that the holographic principle, typically consistent with strong
346 gravitational objects, is potentially involved in the strong field confinement environment of the
347 femtometer scale due to Planck fluctuations. Here our generalized holographic approach has led
348 us to a direct relationship between a cosmological gravitational solution and the Planck scale to
349 produce the mass of a quantum object. From equation (11)

$$m_h = \frac{Rm_\ell}{\eta}$$ (35)

351 where $R$ is the number of PSUs within the interior and $\eta$ is the number of PSUs on the surface
352 horizon, we now clearly discern that both the holographic gravitational mass (equivalent to the
353 Schwarzschild mass) and the rest mass of the proton are a consequence of the Planck mass $m_\ell$,
354 and the geometrical considerations of Planck vacuum oscillators alone.
355
356 Although equation (35) has a simple and elegant form, we now explore a little further the algebra
357 to better understand the geometric relationship between $m_{p'}$, $m_\ell$ and $m_{h'}$.
358

359      Starting from equation (34) and multiplying by $m_{h'}/m_{h'}$ we have

$$m_{p'} = 2\frac{m_\ell^2}{m_{h'}} = 2\frac{m_\ell^2}{m_{h'}^2}m_{h'} . \tag{36}$$

361
362
363
364      Expanding $m_{h'}$ in the denominator with equation (35) and rearranging terms we have

$$= 2\frac{m_\ell^2}{\left(\dfrac{Rm_\ell}{\eta}\right)^2}m_{h'} = 2\left(\frac{\eta m_\ell}{Rm_\ell}\right)^2 m_{h'} . \tag{37}$$

366
367      We now express this in terms of $\eta_\rho$ and $R_\rho$

$$m_{p'} = 2\left(\frac{\eta_\rho}{R_\rho}\right)^2 m_{h'} \tag{38}$$

369      where $\eta_\rho$ is the mass of PSUs on the surface horizon and $R_\rho$ is the mass of PSUs in the interior
370      volume as in equation (10). Here the geometric mass relationship clearly emerges. Significantly,
371      the rest mass of the proton is generated by the square of the simple mass relationship of the
372      surface mass of PSUs to the interior mass of PSUs multiplied by the holographic gravitational
373      mass of the proton. Of course we can also express this relationship in terms of dimensionless
374      quantities. We divide by $m_\ell$ in the numerator and denominator

$$m_{p'} = 2\left(\frac{\eta_\rho/m_\ell}{R_\rho/m_\ell}\right)^2 m_{h'} \tag{39}$$

376      yielding

$$= 2\left(\frac{\eta}{R}\right)^2 m_{h'} . \tag{40}$$

378      Yet, another step can be taken to further elucidate the nature of the relationship by expanding
379      $m_{h'}$ utilizing equation (9)

$$= 2\left(\frac{\eta}{R}\right)^2 \frac{R}{\eta}m_\ell \tag{41}$$

381      which reduces to

$$m_{p'} = 2\frac{\eta}{R}m_\ell \tag{42}$$

383      which can be converted back to a mass only expression by multiplying the dimensionless
384      quantities by $m_\ell$, yielding

$$m_{p'} = 2\frac{\eta_\rho}{R_\rho}m_\ell . \tag{43}$$

386      The relationships between the proton mass, the Planck mass and the holographic gravitational
387      mass clearly emerge from this algebraic sequence of equations. One of the most significant
388      challenges of modern physics has been to find a comprehensive framework to explain the
389      significant discrepancy between the relatively large Planck mass, the mass of the proton, and the
390      gravitational force or what is known as the *hierarchy problem*. Frank Wilczek, whose

391 fundamental contribution of asymptotic freedom to the strong interaction theory, states *"We see*
392 *that the question it poses is not, 'Why is gravity so feeble?' but rather, 'Why is the proton's mass*
393 *so small?' For in natural (Planck) units, the strength of gravity simply is what it is, a primary*
394 *quantity, while the proton's mass is the tiny number..."* [21]
395
396 Here the hierarchy problem between the Planck mass and the proton rest mass is resolved as we
397 clearly demonstrate that the rest mass of the proton is a function of the Planck vacuum oscillators
398 holographic surface to volume geometric relationship of spacetime, the energy levels of which
399 include the gravitational mass-energy $m_{h'}$ derived from the same primary quantity of Planck
400 entities. We express the relationship of the proton surface horizon to its volume Planck
401 oscillators as a fundamental constant we term $\phi$

402
$$\phi = \frac{\eta}{R} = \frac{\eta_\rho}{R_\rho} = 3.839682 \times 10^{-20} \tag{44}$$

403 which appears as a fundamental geometric ratio from equations (38) to (43), whether in
404 dimensionless quantities or in mass ratios. The inverse relationship

405
$$\frac{1}{\phi} = \frac{R}{\eta} = \frac{R_\rho}{\eta_\rho} = 2.604382 \times 10^{19} \tag{45}$$

406 is clearly seen in equation (41) where $m_{h'}$ is fully expanded in its holographic expression from
407 equation (9) of section 2. Therefore, $\phi$ and its inverse relate the gravitational curvature of a
408 Schwarzschild metric to the quantum scale so that

409
$$m_{p'} = 2\phi^2 \frac{1}{\phi} m_\ell = 2\phi^2 m_{h'} \tag{46}$$

410 and relates the proton rest mass to the Planck mass

411
$$m_\ell = \frac{m_{p'}}{2\phi} \tag{47}$$

412 and of course the Planck mass to the holographic gravitational mass is $\phi$

413
$$m_\ell = \phi m_{h'}. \tag{48}$$

414 Consequently $\phi$ acts as a fundamental constant relating the background Planck vacuum
415 fluctuation field to the cosmological and quantum scale where it may be the source of
416 confinement so that scaling from the proton rest mass to the Planck mass requires a proportional
417 mass-energy conversion of $2\phi$ while from the Planck mass to the holographic gravitational mass
418 requires a factor of $\phi$, which yields a total scaling from the proton rest mass to the holographic
419 gravitational mass of

420
$$2\phi^2 = 2.948632 \times 10^{-39}. \tag{49}$$

421 Exploring the $\phi$ relationships relative to quantum gravity confinement, we utilize equation (47),
422 and we determine

423
$$m_{p'} = 2\phi m_\ell = 2\phi \sqrt{\frac{\hbar c}{G}}. \tag{50}$$

424
425 Squaring both sides

426
$$m_{p'}^2 = 4\phi^2 \frac{\hbar c}{G}. \tag{51}$$

427 Multiplying both sides by $\dfrac{G}{\hbar c}$ we have

11

428
$$4\phi^2 = \frac{Gm_{p'}^2}{\hbar c} = \frac{Gm_{p'}m_{p'}}{\hbar c} \ . \tag{52}$$

429 where $4\phi^2 = 5.897264 \times 10^{-39}$ is the exact value for the coupling constant between gravitation
430 and confinement at the proton scale or the strong interaction. The typical computation given for
431 the gravitational coupling constant is

432
$$\frac{F_g}{F_s} = \frac{F_g}{F_e}\frac{F_e}{F_s} = \frac{Gm_p m_p / r^2}{e^2 / r^2}\alpha = \frac{Gm_p^2}{e^2}\alpha = 5.905742 \times 10^{-39} \tag{53}$$

433 where $e$ is the elementary charge and $\alpha$ is the fine structure constant. Note that the slightly
434 different value of equation (53) from $4\phi^2$ of equation (52) is due to our utilization of the new
435 muonic measurement of the radius of the proton, and that utilizing our predicted radius $r_{p'}$ from
436 equation (30) yields the exact value.
437
438 Hence the gravitational force coupling constant is computed directly from the geometric
439 relationship of the Planck oscillator surface tiling to the interior volume oscillations of the proton
440 which as well clearly relate the Planck mass to the proton rest mass, and the $2\phi^2$ ratio of the
441 proton mass to the holographic gravitational mass or the Schwarzschild mass. Consequently, the
442 unifying energy required for confinement is generated by holographic derivations directly from first
443 principles of simple geometric Planck vacuum fluctuation relationships. Furthermore, the rest
444 mass of the proton is computed without requiring the complexities introduced by a Higgs
445 mechanism, which also utilizes a non-zero vacuum expectation value, but which only predicts 1 to
446 5 percent of the mass of baryons, and in which the Higgs particle mass itself is a free parameter
447 [22]. The current QCD approach accounts for the remaining mass of the proton by the kinetic
448 back reaction of massless gluons interacting with the confining color field utilizing special relativity
449 to determine masses. Yet it is critical to note that after almost a century of computation, there is
450 still no analytical solution to the Lattice QCD model for confinement. This problem is thought to
451 be one of the most obscure processes in particle physics and a Millennium Prize Problem from
452 the Clay Mathematics Institute has been issued to find a resolution [23, 24]. Since there is no
453 analytical solution to LQCD and no framework for the energy source necessary for confinement,
454 associating the remaining mass of the proton to the kinetic energy of massless gluons is based
455 on tenuous tenets. Our results demonstrate that the holographic gravitational mass-energy of the
456 proton $m_{h'}$ is the unification energy scale for hadronic confinement and that the mass of
457 nucleons is a direct consequence of vacuum fluctuations. Keeping in mind that a neutron quickly
458 decays into a proton when free of the nucleus, we have therefore addressed the fundamental
459 nature of the nucleon by deriving the proton rest mass and the confining force from holographic
460 considerations. In future publications we will address the confinement string-like gluon jet flux
461 tube structures of the QCD vacuum model as potentially arising from high curvature within the
462 spacetime Planck vacuum collective behavior background, acting as vortices near the
463 holographic screen topological horizon. This will be addressed utilizing an extended center vortex
464 picture which has been significantly developed by 't Hooft [25] and in which the surface area of a
465 Wilson loop is related to a confining force. In the next section, we explore the energy and angular
466 frequency associated with our model and we compute the gravitational potential range of our
467 confining force utilizing special relativity.
468
469 **5. FREQUENCY, ENERGY AND THE YUKAWA POTENTIAL**
470
471 From equations (29) and (47) we have

472
$$m_{p'} = 2\phi m_\ell = 4\ell \frac{m_\ell}{r_p} \ . \tag{54}$$

473 Dividing by $2m_\ell$ on both sides we find

12

$$\phi = \frac{2\ell}{r_p} \tag{55}$$

or

$$r_p = \frac{2\ell}{\phi}. \tag{56}$$

Calculating Einstein's mass-energy equivalence for the proton we have

$$E_p = m_{p'}c^2. \tag{57}$$

From equation (47) we can then write

$$= 2\phi m_\ell c^2 \tag{58}$$

where $m_\ell c^2$ is the Planck energy. Now we expand the terms

$$= 2\phi\sqrt{\frac{\hbar c^5}{G}} = 2\phi\sqrt{\frac{\hbar}{\hbar}\frac{\hbar c^2 c^3}{G}} = 2\phi\sqrt{\frac{\hbar^2 c^2 c^3}{\hbar G}} = 2\phi\hbar c\sqrt{\frac{c^3}{\hbar G}} = \frac{2\phi\hbar c}{\sqrt{\frac{\hbar G}{c^3}}} = \frac{2\phi\hbar c}{\ell}. \tag{59}$$

From equation (56) it follows that

$$= \frac{4\phi\hbar c}{2\ell} = \frac{4\hbar c}{\frac{2\ell}{\phi}} = \frac{4\hbar c}{r_p}. \tag{60}$$

Given that $\hbar = \frac{h}{2\pi}$, then

$$E_p = \frac{4hc}{2\pi r_p} = \frac{4hc}{C_p} = 4hf_p. \tag{61}$$

Thus we have obtained an expression for the energy where $C_p = 2\pi r_p$ is the circumference of

the proton and the angular frequency $f_p = \frac{c}{C_p}$. Therefore the energy of such a system can be

written in terms of $\hbar$ as $E_p = 8\pi\hbar f_p$ which yields a frequency

$$f_p = \frac{E_p}{8\pi\hbar} = \frac{E_p}{4h} = 5.667758 \times 10^{22} hz \tag{62}$$

characteristic to high-energy nuclear gamma emission, and a period of

$$t_p = \frac{1}{f_p} = 1.764366 \times 10^{-23} sec \tag{63}$$

where $10^{-23} sec$ is typically given as the interaction time of the strong force [26]. From equation
(58) we find that $2\phi$ multiplied by the Planck energy yields an angular frequency with a period of
$t_p$, which is the time it takes for a particle to decay via the strong interaction. Hence from the
generalized holographic geometric relations of Planck entities, we have derived clear quantum
gravitational mass-energy formulations that define the characteristics of the strong nuclear force
such as the energies to produce it from gravitational coupling and its interaction time.

13

503 Yet, the short range of the nuclear force as defined by the Yukawa potential demands that the
504 force strength drops off at an exponential rate close to the horizon where $r = r_p$. To explore this
505 force strength to radius relation in our approach, we begin by refining our derivation from
506 reference [27] where we theorize that the difference between the Schwarzschild energy potential
507 and the rest mass of the proton may be the result of mass dilation near the horizon where velocity
508 is relativistic. Therefore, we begin with the known relativistic mass dilation expression

509
$$M = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{64}$$

510 where $m_0$ is a rest mass and $M$ is the dilated mass and $v$ is the velocity. Solving for $\frac{v}{c}$, we
511 find

512
$$\frac{v}{c} = \sqrt{1 - (\frac{m}{M})^2} \tag{65}$$

513
514 Substituting $m_0 = m_p$ and $M = m_h$

515
$$\frac{v}{c} = \sqrt{1 - (\frac{m_{p'}}{m_{h'}})^2} = \sqrt{1 - 4\phi^4} . \tag{66}$$

516
517 Therefore the dilated mass-energy yielding the Schwarzschild unifying energy potential occurs at
518 $\frac{v}{c}$ extremely close to 1. We compute the result and examine how close $v$ is to $c$ and find

519
$$1 - \frac{v}{c} = 4.347214 \times 10^{-78} . \tag{67}$$

520 That is, the Schwarzschild energy potential is reached when $v$ is $4.34 \times 10^{-78}$ less than $c$, which
521 can be computed as well, with an accuracy of some 76 significant digits, to be $2\phi^4$. We now
522 seek an expression for $v$ as a function of $r$ utilizing an orbital velocity formula. Our purpose is
523 to identify velocities at the Schwarzschild horizon or the holographic horizon described in earlier
524 sections. The use of relativistic velocity equations produces results describing velocities at the
525 photon sphere or the ergosphere in the case of the Kerr Metric where the ergosurface is situated
526 at 1.5 times the Schwarzschild radius at the equator (the photon sphere) and is oblate so that the
527 poles are coincident with the Schwarzschild surface. We note that the relativistic photon sphere
528 solution corresponds closely with the Compton wavelength of the proton. However, for our
529 purpose in this work our intent is to compute the velocity at the Schwarzschild surface or
530 holographic surface rather than the ergosphere. For that purpose a simple semi-classical form
531 can be utilized. Therefore

532
$$v(r) = \sqrt{2ar} = \sqrt{2 \frac{Gm}{r^2} r} = \sqrt{\frac{2Gm}{r}} \tag{68}$$

533 and multiplying by $c^2$ in the numerator and denominator and utilizing the Schwarzschild radius
534 equation

535
$$= c\sqrt{\frac{2Gm}{rc^2}} = c\sqrt{\frac{r_s}{r}} . \tag{69}$$

536 Substituting $v(r)$ into the mass dilation equation (64) we have

14

537

$$M = \frac{m}{\sqrt{1 - \frac{[v(r)]^2}{c^2}}} = \frac{m}{\sqrt{1 - \frac{c^2 r_s}{rc^2}}} = \frac{m}{\sqrt{1 - \frac{r_s}{r}}} . \tag{70}$$

538 Substituting $m_{h'}$ for $m$ and $r_p$ for $r_s$, we can derive that the radius at which the unification

539 energy $m_{h'} = 5.668464 \times 10^{14} \, gm$ is achieved due to mass dilation can be computed as

540
$$r = r_p \frac{m_{h'}^2}{m_{h'}^2 - m_{p'}^2} = r_p \frac{m_{h'}^2}{(m_{h'}^2 - (2\phi^2 m_{h'})^2)} = r_p \frac{m_{h'}^2}{m_{h'}^2 (1 - 4\phi^4)} = \frac{r_p}{(1 - 4\phi^4)} \tag{71}$$

541 or the dimensionless quantity $(r - r_p)/r_p = 8.694428 \times 10^{-78}$. Consequently we can assert for

542 all intent and purposes, that the Schwarzschild mass occurs at or extremely close to horizon. We

543 now compute the mass dilation from the velocity found at $\ell$ from $r_p$ utilizing equation 1 and find

544
$$m_{pd}^\ell = \frac{m_{p'}}{\sqrt{1 - \frac{r_p}{r_p + \ell}}} = \sqrt{\frac{2}{\phi} \frac{(r_p + \ell)}{r_p}} m_{p'} = 1.206294 \times 10^{-14} gm \tag{72}$$

545 where $m_{pd}^\ell$ is the dilated mass at one Planck length from $r_p$. Evidently an asymptotic drop of the

546 dilated mass-energy $m_{h'}$ occurs, reducing by some 28 orders of magnitude within one Planck

547 length from the horizon. We note that $\dfrac{m_{pd}^\ell}{2}$ is equivalent to the geometric mean $\sqrt{m_{p'} m_{h'}}$

548 between the Planck mass and the rest mass of the proton, which may represent a harmonic

549 relationship between $m_{pd}^\ell$ and $m_{p'}$.

550
551 We now utilize equation 1 to compute mass dilation as a function of radius, which we convert to a
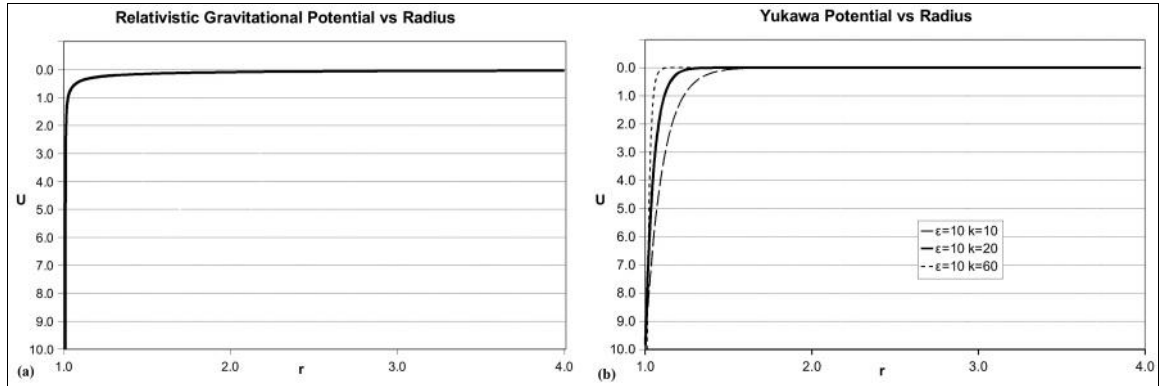
552 gravitational energy potential $Gm/r$. We graph our results and compare them with the Yukawa

553 potential, see figure 1a.

554



555
556 **Figure 1.** *(a) The relativistic gravitational potential ∪ resulting from mass dilation near the horizon $r_p$. (b)*

557 *The Yukawa potential ∪ typically given as the short range energy potential of the strong force where $\varepsilon$ is*

558 *the hard-core surface potential and $k$ is the inverse screening length (inverse Debye length).*

559
560 From Figure 1(a) we find that the gravitational potential from the mass dilation of a proton due to

561 the angular velocity of an accelerated frame generates an asymptotic curve with a force potential

562 drop-off as a function of $r$ characteristic of the short range force of nuclear confinement

563 equivalent to the Yukawa potential in figure 1(b). Therefore, we have derived a relativistic source

15

564  for the confining energy with a quantum gravitational potential equivalent to the unification energy
565  of a Schwarzschild mass or the holographic gravitational mass of the proton $m_{h'}$, yielding a
566  gravitational coupling with a Yukawa-like short range, and the appropriate interaction time of the
567  strong force $t_p$, resulting in an analytical solution to confinement. These results are derived from
568  first principles and classical considerations alone, with zero free parameters or hidden variables,
569  and extend our generalized holographic solution to generate a complete picture of confinement
570  whether at the quantum scale or the cosmological scale of black holes. Furthermore,
571  considerations of equations (38) and (43), where the rest mass of the proton is derived from
572  relationships of Planck oscillators PSUs of an energetic structured vacuum at the holographic
573  horizon, may provide us with a source for mass. This is analogous to the non-zero vacuum
574  expectation value of the Higgs field where the Yukawa interaction describes the coupling between
575  the Higgs mechanism and massless quark and lepton fields or fermions. However, this Higgs
576  mechanism only accounts for a small percentage of the mass of baryons where the rest is
577  thought to be due to the mass added by the kinetic energies of massless gluons inside the
578  baryons. Our generalized holographic model accounts for all of the rest mass of protons and the
579  energy of confinement in addition to predicting the mass of cosmological objects directly out of
580  geometric considerations of the energetic vacuum.
581
582  **6. CONCLUSION**
583
584  We have generalized the holographic principle to considerations of spherical tiling of
585  Planck vacuum fluctuations within volumes as well as on horizon surfaces. From these
586  discrete spacetime quantization relationships we extract the Schwarzschild solution to
587  Einstein's field equations, generating a novel quantized approach to gravitation. We
588  apply this resulting quantum gravitational method to the nucleon to confirm its relevance
589  at the quantum scale and we find values for the rest mass of the proton within
590  $0.069 \times 10^{-24} gm$ or ~4% deviation from the CODATA value and $0.0012 \times 10^{-24} gm$ or
591  ~0.07% deviation when the recent muonic radius measurement is utilized. As a result,
592  we predict a precise proton charge radius utilizing our holographic method which falls
593  within the reported experimental uncertainty for the muonic measurement of the proton
594  charge radius. More precise experiments in the future may confirm our predicted
595  theoretical proton charge radius.
596
597  We determine a fundamental constant $\phi$ defined by the mass ratio of vacuum
598  oscillations on the surface horizon to the ones within the volume of the proton. As a
599  result, clear relationships emerge between the Planck mass, the rest mass of the proton,
600  and the Schwarzschild mass of the proton or what we term the holographic gravitational
601  mass. Furthermore, we find that $4\phi^2$ generates the coupling constant between
602  gravitation and the strong interaction, thus defining the unification energy for
603  confinement. We also derive the energy, angular frequency, and period for such a
604  system utilizing our holographic approach and find that the frequency is the
605  characteristic gamma frequency of the nucleon and the period is on the order of the
606  interaction time of particle decay via the strong force. Finally, we calculate the mass
607  dilation due to velocity as a function of radius and plot the resulting gravitational potential
608  range. We find the range to be a close correlation to the Yukawa potential typically
609  utilized to illustrate the sharp drop-off of the confining force. In future work we will
610  examine the application of this approach to more complex systems. We will consider as
611  well some of the seminal work done in defining maximal particle momentum and it's
612  applicability to our approach [28].
613

In this paper, we demonstrate that a quantum gravitational framework of a discrete spacetime defined by spherical Planck vacuum oscillators can be constructed which applies to cosmology and quantum scale. Our generalized holographic method utilizes zero free parameters and is generated from simple geometric relationships and algebra, yielding precise results for significant physical properties. In the words of Einstein, *"One can give good reasons why reality cannot at all be represented by a continuous field. From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers (quantum numbers). This does not seem to be in accordance with a continuum theory and must lead to an attempt to find a purely algebraic theory for the representation of reality."* [29]

## ACKNOWLEDGEMENTS

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