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Geometric Phase, Curvature, and the Monodromy Group

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ABSTRACT

AimsThe aim of this paper is to show that geometric phase is a consequence of curvature in non-Euclidean geometries, being related to the areas of spherical and hyperbolic triangles. In hyperbolic geometry it is well-known that the angular deficit of a hyperbolic triangle is related to Wigner rotation and Thomas precession, whereas in spherical geometry, its relation to automorphic functions arising from Fuchsian differential equations containing non-essential singularities has not been appreciated. It is the aim of this paper to fill this lacuna.

Methodology: Fuchsian differential equations with non-essential singularities are solved by a power series solution (indicial equation) and the quotient of two solutions will undergo linear-fraction transformations which tessellate the half-plane or unit disc with curvilinear triangles or lunes depending on the number of singular points. Their inverse aremultivalued, periodic, or automorphic, functions. Analytic continuation about a singular point does not give back the original solution. Multivaluedness is the cause of geometric phase. Examples are: the Pancharatnam phase of beams of polarized light, the Aharonov-Bohm effect, the Dirac monopole, and angular momenta with 'centripetal' attraction in the case of spherical geometry. These will be compared with non-collinear Lorentzian boosts that are responsible for Wigner rotation and Thomas precession in hyperbolic geometry, where the angle defect is related to the Euclidean measure of hyperbolic distance of two sides of a hyperbolic triangle in velocity space. For a right hyperbolic triangle, the angular defect isthe angle of parallelism.

Results:A finite geometric phase requires non-integral quantum numbers, and, thus cannot be associated with 'particles.' By conformal transformation, the homologues of the poles can be transformed into vertices of lunes, curvilinear triangles, and polygons which place restrictions on the range of angular momenta. In contrast to quantum mechanics, where space is continuous and quantum numbers discrete, the space is now discrete, made up of tessellations which are repetitions of the fundamental region without lacunae and without overlap, and the interval of the quantum numbers is continuous.

Conclusion: Many of the equations of mathematic physics can be reduced to second-order Fuchsian equations with real coefficients in the limit of vanishing kinetic energy where essential singularities are reduced to simple poles. For only then will the solutions to the differential equations be rational functions in order that the covering group will be cyclic, and the covering space will be a 'spiral staircase, like the different leaves of a Riemann surface.

Keywords: [geometric phase, non-Euclidean geometries, Gaussian curvature, holonomy, multivaluedness, Aharonov-Bohm effect, Dirac monopoles, automorphic functions,monodromy group, Fuchsian differential equations, Wigner rotation]

1. INTRODUCTION

Quantum mechanics goes to great lengths to ensure that the wave functions are single-valued. This means discarding terms in the solution of the Schrödinger equation that either blow up at the origin or diverge at infinity. Solutions of second-order differential equations which are rational lead to multivaluedness, and great efforts were spent, in the late nineteenth century, to uniformize the solutions so as to render them single-valued. However, multivaluedness is not a stigma, and it will explain numerous phenomena from the interaction of polarized beams to the Aharonov-Bohm effect. In this paper we treat multivaluedness from the theory of automorphic functions.

If a vector is parallel-transported around a closed curve it may not necessarily return as the same vector it started out as. The effect is known as holonomy, and it has been attributed to positive, Gaussian curvature [1].Holonomy also occurs when we solve a Fuchsian differential equation as a power series, and analytically continue around a singular point. We will not, in general, get back the solution we started with, but one that differs from it by a phase factor.

We will show that geometric phase is a manifestation of periodicity with respect to a group of motions of the tessellations of a disc, or half-plane, by lunes or curvilinear triangles, depending on whether the Fuchsian differential equation has two or three regular singular points, respectively. Functions whose only singular points are rational functions will be solutions to a Fuchsian differential equation of two singular points, while the solutions of one with three regular points will not reduce to elementary functions, but, rather, can be expressed as a Euler beta integral.

Differential equations containing only regular singular points, like the hypergeometric equation, have very little to do with equations of mathematical physics [2]. Although equations of mathematical physics have a regular singular point at the origin, they possess an essential singularity at infinity that prevents the solution from diverging at infinity. The regular singular point at the origin has two linearly independent solutions, which are powers of the radial coordinate whose exponents are determined by the roots of the indicial equation. Their quotient is an automorphic function, whose inverse is a periodic function that will undergo a linear-fractional transformation, whose motion will tessellate the plane with lunes, or curvilinear triangles without overlap or lacunae. Quantum mechanics eliminates one of these solutions on the basis that it blows up at the origin, and, hence, is unphysical. This eliminates the possibility of constructing automorphic functions as quotients of the two independent solutions of the indicial equation.

Because the kinetic energy is finite, the other singularity at infinity is an essential singularity. The solutions are exponential rising and decaying functions of the radial coordinate. In order that the wave function be finite and single-valued, the rising solution is eliminated. The essential singularity arises from a coalescence of two regular singular points, and it is analogous to the behavior of an automorphic function in the immediate neighborhood of limit points of the group of motions which tessellate the half-plane, or unit disc. Consequently, if we allow for multi-valuedness of the Schrödinger equation, its solutions will behave like automorphic functions far from the limit points on the boundary when we consider the limit of vanishing kinetic energy.

In the next three sections we will argue that geometric phases requires positive Gaussian curvature so that the ratio of the area of a curvilinear triangle to its angular excess is constant. We will do so through a detailed discussion of thephasor, thePancharatnam phase of polarized light beams, the Aharonov-Bohm phase and the Dirac monopole. Periodicity is with respect to a group of motions which tessellate the half-plane, or disc, which have natural boundaries, the real half-line and the principal circle, respectively, along which the essential singularities lie. Periodicity requires at least two regular points, and the elliptic motion is a rotation. Non-integral quantum numbers are required in order that the group not reduce to the identity, corresponding to the equivalence class of null paths. As such, non-integral quantum numbers do not represent particles, whose quantum numbers must be integers, but, rather, should be considered as resonances.

We then discuss 'centripetal attraction,' where the angular momentum varies over a continuous range of non-positive, non-integral values. The quotient of the solutions to the differential equation will take on each value only once in the fundamental region, which is a lune. This forms a dichotomy with quantum mechanics, where the angular momenta are discrete and space is continuous. Now, the angular momenta are continuous and space is discrete. We then go on to reconstruct the original Schrödinger equation: for negative kinetic energy the essential singularity is an exponential function, while for positive kinetic energy it is a circular function. As long as the kinetic energy is zero, the Schrödinger equation, even in the presence of a potential, can be reduced to a Fuchsian form with multiple space scales.

We conclude the paper with a comparison of geometric phase in hyperbolic geometry. Although Wigner rotation and Thomas precession are known examples of geometric phase, we relate the angular defect of a hyperbolic triangle to the Euclidean measures of the sum of the lengths two sides of a hyperbolic triangle. In the case of a right triangle,the angular defect coincides with the angle of parallelism discovered by Lobachevsky and Bolyai.

2. PHASOR AND THE CONSTRUCTION OF AN ESSENTIAL SINGULARITY

The linear-fractional transform,

$$w=\frac{az+b}{cz+d}, \tag{1}$$

guaranties that the fundamental region will have the same number of poles and zeros, where a, b, c, and d are constants such that $ad-bc=1$. The difference between the number of zeros, n, and the number of poles, p, is

$$\frac{1}{2\pi i}\oint\frac{f'(z)}{f(z)}dz = n-p, \tag{2}$$

where the contour encloses all zeros and poles. Setting $f(z)=w$, where w is given by (1), we find

$$\frac{1}{2\pi i}\oint\left(\frac{1}{z+\frac{b}{a}}-\frac{1}{z+\frac{d}{c}}\right)dz = 0. \tag{3}$$

Multiple moments of order m,

$$\frac{1}{2\pi i}\oint z^m\frac{f(z)}{z-a}dz, \tag{4}$$

are analogues of essential singularities [3]. Since (4) vanishes for an automorphic function, there can be no concentration of 'charges.' Charges are the analogs of zeros and poles, and equation (3) expresses charge neutrality.

For real values of the coefficients in (1), the zeros will fall on the real axis. The contour in the z-plane for the linear-fractional transformation (1) is a circle through the pole at $-d/c$, and zero, $-b/a$, as shown in Fig.1. The phase, δ, at point P, is the difference between the angle β and the exterior angle α [3]

$$\delta = \beta - \alpha. \tag{5}$$

Lines of constant phase are circles passing through $-b/a$ and $-d/c$.

The crucial, and new, realization is that by adding δ to both sides of (5), and adding and subtracting π on the right-hand side give

(6) $$2\delta = \delta + \beta + (\pi - \alpha) - \pi > 0.$$

The right-hand side of (6) is precisely the anglular excess of a spherical triangle. We will appreciate in the next section that the phasor, (5), is the complementary angle to the Pancharatnam phase (12) below.

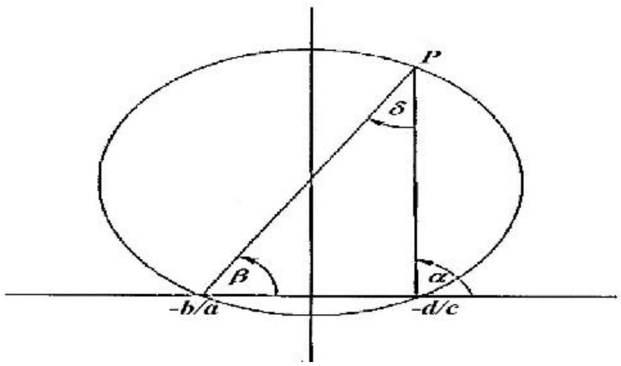

Fig.1 Thecontour is a circle passing through the pole at $-d/c$ and the zero $-b/a$.

The three angles of the triangle in Fig. 1, $\delta = \lambda\pi$, $\beta = \mu\pi$, and $\pi - \alpha = \gamma\pi$, correspond to three *regular* singular points, which by a linear-fractional transformation can be placed at $0, 1$, and ∞, for convenience. Any three points can be transformed into any three other points, which form a complete set of invariants. The simplest Fuchsian differential equation whose solutions do not reduce to elementary rational functions is one with three singular points. With β at the origin, $\pi-\alpha$ at 1, the phasorδ will be found at infinity.

The automorphic function,

(7) $$\mathrm{w} = \int^z z^{\mu-1}(1-z)^{\gamma-1}dz,$$

is a Euler beta integral, and it satisfies the Fuchsian differential equation of second-order,

(8) $$w'' = \left(\frac{\mu-1}{z} + \frac{1-\gamma}{1-z}\right)\ \mathrm{w}',$$

where the primes stand for differentiation with respect to z. The value of the third angle, δ, at infinity can be determined from the Schwarzian derivative,

$$\{\mathrm{w}, \mathrm{z}\} = \frac{1-\mu^2}{2z^2} + \frac{1-\gamma^2}{2(1-z)^2} + \frac{(1-\gamma)\ \ (1-\mu)}{z(1-z)}.$$

Equating the numerator of the last term with the canonical form [4],

$$\gamma^2 + \mu^2 - \lambda^2 - 1 = -2(1-\gamma)(1-\mu),$$

results in

(9) $$\lambda = \pm\,(\gamma + \mu - 1).$$

The negative sign will give the Euclidean result,

(10) $$\pi = \delta + \pi - \alpha + \beta,$$

which is the *negative* of the phasor (5), while the positive root in (9) will give the correct phasor, (5). This proves that the phasor belongs to spherical geometry, and not to Euclidean geometry as previously believed.

3. PANCHARATNAM'S PHASE FOR POLARIZED LIGHT

Berry [5] claimed that Pancharatnam's phase [6] is one-half the solid angle subtended by a geodesic triangle on the Poincaré sphere. Without any knowledge of what the Pancharatnam phase is, it can safely be ruled out that the phase would be related to an interior solid angle when it is well-known that all deductions are made on the surface of the

Poincarésphere with absolutely no knowledge of the interior angles or points that the sphere encompasses [7]. Moreover, any shape on the surface of the sphere that has the same area will have the same solid angle, and, consequently, it need not be a geodesic triangle. In contrast, we will show that the complementary angle found by Pancharatnam is equal to one-half the area of a spherical triangle, as given by the angle excess.

Pancharatnam considered a polarized beam C to be split into two beams in states of polarization A and B, and whose phase difference is the complementary angle to δ. In regard to the phasor (5), δ will be equal to the difference in the internal angle $\angle ACB$ and the exterior angle $\angle ABC'$,

(11) $$\delta = \angle ACB - \angle ABC',$$

as shown in Fig. 2. Expressing the exterior angle in terms of the interior angle, and adding $\delta = \angle BAC$ to both sides of (11) give

(12) $$2\delta = \angle BAC + \angle ACB + \angle ABC - \pi.$$

Equation (12) expresses twice the phase difference between two beams in terms of the area of a spherical triangle given by the angle excess.

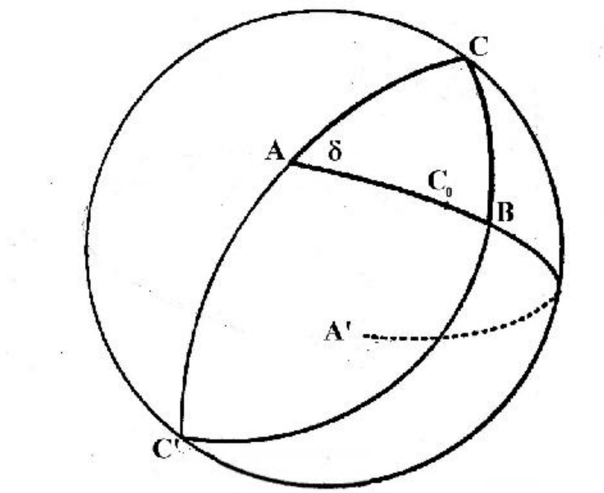

Fig. 2 The phase $\angle C'AB$ is determined by the angle excess of the triangle ΔBAC columnar to $\Delta C'AB$. As $B \rightarrow C$, the two beams will have opposite phases, while as $B \rightarrow C'$, which is the opposite state of polarization to C, the phase will vanish.

Actually, Pancharatnam defines $\delta = \angle CAB$ as the phase difference which he expresses in terms of the triangle columnar to $\triangle ACB$, i.e., $\triangle AC'B$. In other words, the angle,

(13) $$\angle C'AB = \angle AC'B - \angle ABC,$$

is the phasor (5), being the difference between the opposite internal angle and the external angle of the third angle of the spherical triangle. Adding the angle $\angle C'AB$ to both sides of (13), and adding and subtracting π on the right-hand side yield:

(14) $$2\angle C'AB = \angle C'AB + \angle AC'B + \angle ABC' - \pi.$$

The right-hand side of (14) is the area of the triangle $\triangle C'AB$, and replacing the left-hand side by its complementary angle gives

(15) $$\delta = \angle CAB = \pi - \tfrac{1}{2}(\angle C'AB + \angle AC'B + \angle ABC' - \pi),$$

which is equation (5.a) in Pancharatnam [6].

As $B \to C$, the phase $\angle C'AB' \to \pi$, and the beams will have opposite phases. This is analogous to the coalescence of the zero and pole to form a multipole. Alternatively, as $B \to C'$, the opposite state of polarization to C, the beams in the states of polarization A and B will have zero phase difference.

Pancharatnam then asked what happens when the split component B tends to the opposite polarized state A' of the other polarized component of A? As $B \to A'$ and $\delta \to \triangle$, where

(16) $$\triangle = \pi - \angle C_0AC' = \angle C_0AC,$$

δ will be given in terms of the area of the lune cut out by the great circles AC_0A' and $AC'A'$, or $2\angle C_0AC'$.

Figure 2 also illustrates Pancharatnam's observation that the emergent state of polarization C can be obtained from the incident state of polarization C_0 when polarized light passes through a birefringent medium. This can be viewed as a rotation of the Poincaré sphere through and angle $\triangle$ in the counterclockwise direction about the AA' axis.

4. THE AHARONOV-BOHM EFFECT

The fringe shift in a field free, but multivalued, region due to a non-vanishing vector potential was predicted by Ehrenberg and Siday [8], and rediscovered a decade later by Aharonov and Bohm [9].It consists in a two-slit diffraction phenomenon in which the magnetic field is confined to the interior of the solenoid placed in between the slits. Although the particles passing through the slits never pass into a region of non-zero magnetic field, as the flux in the solenoid is increased from zero, the phase of the path that goes through the upper slit changes in respect to the phase of the path going through the lower slit so that a diffraction pattern is produced although neither particle experiences a magnetic field. To explain such an effect, Aharonov and Bohm insisted on the multivaluedness of the regions in which the beams are travelling.

The problem is closely allied to the existence of a magnetic monopole, first postulated by Dirac [10]. Dirac's prescription was to write the wave function as a product of a field free wave function, ψ_0, and a phase,

(17) $$\psi(r,t) = \psi_0(r,t)e^{ie\int \boldsymbol{A}\cdot d\boldsymbol{r}},$$

in units where $c = \hbar = 1$, where $\boldsymbol{A}$ is the vector potential and e is the electric charge. One would expect that (17) would satisfy the Schrödinger equation

(18) $$i\frac{\partial\psi}{\partial t} = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2\psi = \frac{1}{2m}\{-1/r^2\ \partial/\partial r\, r^2\, \partial/\partial r + 1/r^2\ [\boldsymbol{r} \times (\boldsymbol{p} - e\boldsymbol{A})]^2\}\,\psi.$$

But, since the angular momentum,

(19) $$\boldsymbol{L} = \boldsymbol{r} \times (\boldsymbol{p} - e\boldsymbol{A}),$$

equation (18) is independent of the vector potential $\boldsymbol{A}$.In other words, (18) depends on the angular momentum (19) whatever be its origin. So what is the significance of the phase factor in (17) when (18) is effectively independent of A?

If the integral in the phase is intended as a closed circuit then by Stokes' law it is equal to themagnetic flux, Φ, through the surface. And if the magnetic flux is replaced by a monopole of strength g, then (17) will be multivalued unless $eg$is an integer. This is Dirac's quantum prescriptionfor the quantization of electric charge. The presence of a single monopole will lead to the quantization of charge. So it is not the potential that has to be treated as a physical field, and which is also directly observable [11]. This also means that the shift in the diffraction pattern is also independent of the choice of gauge of the vector potential [12].

For suppose that $A$is observable. By a phase factor, the left-hand side of (18) can be reduced to $E\psi$, and with $\boldsymbol{p} = -i\boldsymbol{\nabla}$, the radial equation becomes

(20) $$\psi'' + P\psi' + Q\psi = E\psi,$$

where the prime denotes differentiation with respect to r and

$$P = -2ieA, \text{and}\ Q = -(ieA' + e^2A^2)\ .$$

Now we want to get curvature out of (18), and in order to do so the energy must be subtracted from the Hamiltonian "which produces a trivial, computable phase change in the solution" [13]. This is not trivial, however, since a finite energy would bring in higher-order poles in the indicial equation, and would introduce an essential singularity into the Schrödinger equation.

Roughly speaking, the Schwarzian derivative, or Schwarzian for short, means curvature [14], and we want to transform (20) into a form where the Schwarzian manifests itself. Two systems are said to be strongly equivalent[1] if a change in the unknown $\psi \to \kappa\psi$ transforms one into the other. Under this transformation, and with the zero-kinetic energy condition, (20) becomes

(21) $\psi'' + \left(P + 2\frac{\kappa'}{\kappa}\right)\psi' + \left(Q + P\frac{\kappa'}{\kappa} + \frac{\kappa''}{\kappa}\right)\psi = 0.$

If κ satisfies (20) with $E = 0$, then the coefficient of ψ vanishes in (21). But, the surviving coefficient would not be an invariant because it is independent of Q. Rather, if we choose κ so that the coefficient of ψ' vanishes, i.e., $\kappa'/\kappa = -\frac{1}{2}P$, then we find

(22) $$Q + P\frac{\kappa'}{\kappa} + \frac{\kappa''}{\kappa} = Q - \tfrac{1}{4}P^2 - \tfrac{1}{2}P'.$$

This is exactly the Schwarzian which is found to vanish identically. Hence*, the vector potential does not introduce curvature, ormultivaluedness*. Multivaluedness is rather to be associated with monodromy, or the failure to be single-valued as we 'run around' a path encircling the singularity.

Wu and Yang [16] modified the angular momentum (19) to read

(23) $\boldsymbol{L} = \boldsymbol{r} \times (\boldsymbol{p} - e\boldsymbol{A}) - \alpha\frac{\boldsymbol{r}}{r},$

so that the square of the angular momentum,

$$[\boldsymbol{r} \times (\boldsymbol{p} - e\boldsymbol{A})]^2 = L^2 - \alpha^2,$$

has a negative contribution. In contrast, Aharonov and Bohm introduced the square of the z-component of the angular momentum,

(24) $L_z = e^{i\alpha\theta}\left(-i\frac{\partial}{\partial\theta}\right)e^{-i\alpha\theta} = -\left(i\frac{\partial}{\partial\theta} + \alpha\right),$

which also has a negative component to get

(25) $\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\theta} - ia\right)^2 + k^2\right]\psi = 0,$

where$E = \frac{k^2}{2m}$, outside the region of the magnetic field, and**k** is the wave vector of the incident particle. Since the solution to the radial equation, found by Tamm [17], is a Bessel function, $\frac{J_\mu(kr)}{\sqrt{kr}}$, with index,

$$\mu = \sqrt{[l(l+1) - \alpha^2 + \tfrac{1}{4}]},$$

Wu and Yang required$l(l+1) \geq \alpha^2$, or more precisely, $l + \tfrac{1}{2} > \alpha$.

According to Wu and Yang [16], equation (18) has no meaningful solution for $k^2 \leq 0$. However, it is precisely the equality that allows (18) to be transformed into the Fuchsian differential equation,

(26) $\psi'' + \frac{1-4a^2}{4r^2}\psi = 0,$

provided$2\alpha < 1$, through the transformation $\psi \to \psi/\sqrt{r}$.Wu and Yang argued that since the space around a monopole is spherically symmetric without singularities, the wave function of the electron about the monopole should possess no singularities. Simon [13] contended that holonomy results from a non-real Hamiltonian caused by magnetic fields, or some similar type of phenomenon. It will be clear from our presentation, that analytic continuation about a regular singular point gives rise to geometric phase which is due to an 'attractive' centripetal potential, or a monopole in the expression for the angular momentum (23).

Equation (26) is valid about the singular point at the origin, as well as the singular point at infinity. This can easily be verified by making the substitution$r = 1/z$ in (26) to get

[1] Two sets of linear differential equations are said to be weakly equivalent if one is converted into another when the same transformation is applied to the unknowns as well as the independent variables. They are said to be strong equivalent when one set is transformed into the other when the transformation is applied only to the unknowns [15].

(27) $\psi'' + \frac{2}{z}\psi' + \frac{1-4a^2}{z^2}\,\psi = 0.$

Then, the substitution $\psi \to \psi/z$ will bring it into the exact same form as (27). This shows that the singular points at r=0 and r=∞ are both symmetrical and regular.

The two independent solutions to (27) are:

(28) $\psi_1 = r^{(1+2\alpha)/2}$, and $\psi_2 = r^{(1-2\alpha)/2}$.

Since (28) are multivalued, one solution would have to be rejected to preserve the single-valuedness of the Schrödinger wave function. The quotient of the two solutions, (28), will undergo a linear-fractional transformation since any two independent solutions are linear combinations of any other pair of solutions.

Analytic continuation about the origin, or infinity, will not give back the solution that we started with. The solutions (28) are automorphic functions with respect to the group of rotations. The group tessellate the upper half-plane, or unit disc, by lunes, of the form shown in Fig. 3, where r=0 and r=∞ correspond to the angular points of the lune.

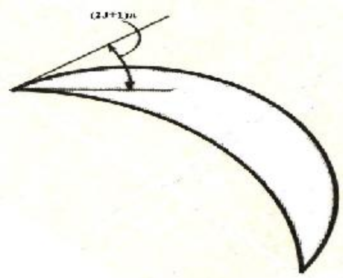

Fig. 3 Two circular arcs intersect at an angle $2\alpha\pi$.

Two circular arcs that cut out the lune intersect at an angle $2\alpha\pi$. The area of the lune is twice that, $4\alpha\pi$. In terms of the phasor, the phase angle would be half this angle, while Panacharatnam gives the phase as the complementary angle. Since we want the phase to vanish with the magnetic flux intensity, we choose the former and get

(29) $\delta = 2\pi\alpha = 2\pi e\Phi.$

The phase factor,

(30) $\psi = e^{i2\pi e\Phi}$,

is the change in the wave function during a circuit around the solenoid. Equation (30) says that when Φ is an odd multiple of a fluxion, $(2e)^{-1}$, the two beams (one bypasses the toroidal magnetic and the other pass through its hole) should exhibit a (maximum) phase difference of π (mod 2π), i.e.

(31) $\frac{2\nu+1}{2}\,2\pi \equiv \pi$ (mod2π) ν = 0, ±1, ±2, ...

This is what is observed in the interferogram that results from the combining the beam with the coherent reference beam that avoids the magnetic field [18]. It is seen that integral quantization of the phase eliminates the phase factor (30) altogether.

Denote by $\lfloor\alpha^{-1}\rfloor$ Gauss' bracket, which indicates the largest integer not exceeding α^{-1}. Then $\varepsilon=\exp(2\pi i/\lfloor\alpha^{-1}\rfloor)$ is an elliptic generator with period $\lfloor\alpha^{-1}\rfloor$ .In other words, there will be $\lfloor\alpha^{-1}\rfloor$ distinct branches, or $\lfloor\alpha^{-1}\rfloor$ 'steps' in the 'spiral staircase.' The different branches are $g_n=\varepsilon^n g_0$ where n=0, 1, 2,..., $\lfloor\alpha^{-1}\rfloor$-1 are the winding numbers. Each step can be regarded as a covering space corresponding to a particular branch of the multivalued function. In particular, for destructive interference of the beams, $\lfloor \alpha^{-1} \rfloor=2$, so that there is a single branch, and the surface is simply connected.

5. ATTRACTIVE ANGULAR MOMENTUM

Many of the equations of mathematical physics can be transformed into Fuchsian differential equations at vanishing kinetic energy. Consider the spherical Bessel function:

(32) $\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{l(l+1)}{r^2}\psi + k^2\psi\right] = 0.$

Bessel'sdifferential equation, (32), has a regular singular point at r=0, and an essential singularity at r=∞. This can easily be checked by substituting z=1/r, and noting that the coefficient of ψ has higher-order poles at z=0.

The indicial equation at the regular point, z=0, has two independent solutions:

(33) $\psi_1 = r^{l+1}$ and $\psi_2 = r^{-l}$.

The second solution, ψ_2, is usually discarded on the basis that it blows up at the origin. Although this makes ψ single-valued, we will not follow this practice. Rather, we form the quotient of the two independent solutions,

(34) $s = \frac{\psi_1}{\psi_2} = r^{\lambda}$,

which is multivalued and automorphic with respect to the group of rotations that will tessellate the half-plane, or disc, with lunes, if and only if $k^2=0$. This is to say that there can be no constant term appearing in the Schwarzian derivative [cf. equation (38) below], *and* $l(l+1) < 0$.

When $k^2 \neq 0$ there will be an essential singularity at r=∞. We may study this singularity by making the usual substitution, z=1/r, and as $z \to 0$, equation (32) will reduce to

(35) $\psi'' + \frac{2}{z}\psi' + \frac{k^2}{z^4}\psi = 0.$

The solution to (35) gives an essential singularity,

(36) $\psi = \sin(k/z)$,

atz=0 consisting of a pole of infinite order. It is the limit point of two sequences of zeros, one on the positive real axis, and the other on the negative real axis [3]. Since the integrand of (2) is

(37) $\frac{f'(z)}{f(z)} = -\frac{k}{z^2}\cot\frac{k}{z} = -\frac{1}{z} + \frac{k^2}{3z^3} + \frac{k^4}{45z^5} + \cdots,$

and introducing it into (4) shows that it has a 'charge' of -1, a vanishing dipole moment, a quadrupole moment of $\frac{1}{3}k^3$, a hexadecapole moment of $k^4/45$, etc.

The automorphic function(34) has the Schwarzian,

(38) $\{s,r\} = \frac{1-\lambda^2}{2r^2} \equiv 2I,$

only in the case of vanishing kinetic energy, $k^2=0$, where $\lambda=2l+1$. As we have already shown, the indicial equations will then be identical about r=0 and r=∞, thereby reducing the second singular point from an essential to a regular singular point. This is necessary insofar as the analytic continuation of the solution about the singular point will not give back the solution that we started out with, but, rather the product of analytic continuations about two singular points will give back the original solution. In the case of two singular points, the generators will be inverse of one another. This is Riemann's condition for the "periodicity of the function" [2], and the group generated by these matrices is the 'monodromy' group, a term coined by Jordan. The monodromy group is a group of transformations that fail to be single-valued as we 'run round' a paththat encircles the singularity.

When the two poles are regular, a simply closed circuit in the counterclockwise direction about r=0, described by the monodromy matrix,

(39) $\mathbf{S_0} = \begin{pmatrix} e^{2\pi i l} & 0 \\ 0 & e^{-2\pi i l} \end{pmatrix},$

must be accompanied by a counterclockwise circuit about the other singular point at infinity,

(40) $\mathbf{S}_\infty = \begin{pmatrix} e^{-2\pi i l} & 0 \\ 0 & e^{2\pi i l} \end{pmatrix}$,

in order that Riemann's condition,

(41) $\mathbf{S}_0\mathbf{S}_\infty = \mathbf{I}$,

be fulfilled. The motions form a group--- the monodromy group. Periodicity results in a multivalued function only for non-integral values of l. Integral values would reduce the monodromy matrices, (39) and (40), to the identity matrix, **I**, and destroy the tessellations of the half-plane, or unit disc by lunes, just like integral values of the magnetic flux would make the shift in the diffraction pattern disappear in the Aharonov-Bohm effect.

This is the condition for constructive inference, which is no longer possible when the singular point at infinity becomes an essential singularity. The presence of the essential singularity destroys the periodicity with respect to themonodromy group. The existence of a lune formed from two circular arcs with angles $\lambda\pi$ implies that $\lambda \leq 1$, or, equivalently, $l \in [-½, 0]$. The centripetal repulsion, $(l+1) > 0$, has now become centripetal 'attraction' $(l+1) < 0$.

Bessel's differential equation (32) thus becomes identical with the Aharonov-Bohm equation, (25). Theautomorphic function, (34), can be written more generally as:

(42) $$S = \frac{as+b}{cs+d},$$

which gives a conformal representation of the S-lune upon the s-half plane. Inside the lune, which is the fundamental region, the automorphic function takes on any value only once. Thus, the linear-fractional transformation (42) will transform two circles cutting at an angle $\lambda\pi$ into any two others interesting at the same angle. This result has been known since the time of Kirchhoff [19].

Thus, space and angular momentum have switched their characteristics: the former is now discontinuous while the latter is continuous in the closed interval [-½,0]. The geometric phase is now half the area of the lune, $\delta=(2l+1)$. For l=-½ the regular and irregular solutions, (33), coalesce and the phase vanishes. At the other extreme, l=0, the geometric phase $\delta=\pi$,for which the area of the lune becomes the area of a hemisphere, and the Schwarzian derivative (38) vanishes. Bessel's differential equation (32) becomes weakly equivalent to $\psi''=0$ so that there is no invariant (38), exactly as in the case of the Schrödinger equation (18).

6. RECONSTRUCTION OF THE SCHRÖDINGER EQUATION

For Fuchsianautomorphic functions, accumulation, or limit, points occur on the principal circle, or the real axis of the half-plane [20]. Not all points on the boundary are limit points of the group. If the automorphic function is not a constant, each limit point of the group is an essential singularity of the function. The behavior of the automorphic function at a limit point is analogous to the behavior of the Schrödinger equation in the immediate neighborhood of point at infinity. In this section, we first establish the form of the essential singularity in the case of negative kinetic energy,[2] and then show that the Schrödinger equation can be reduced to Fuchsian form even in the presence of a potential at infinity provided the kinetic energy vanishes.

Consider the radial Schrödinger equation for *bound* states of the hydrogen atom:

(43) $\psi'' - \left[\frac{l(l+1)}{r^2} - \left(\frac{\gamma}{r} - ¼\right)\right]\psi = 0$,

where the parameter, $\gamma = 1/k\, r_B$,with r_B is the Bohr radius. As $r \to 0$(40) becomes [cf. equation (26)]:

(44) $$\psi'' - \frac{1-\lambda^2}{r^2}\psi = 0,$$

which has has two independent solutions (33),while, as $r \to \infty$, (43) transforms into

(45) $$\psi'' + \frac{2}{z}\psi' - \frac{1}{4z^4}\psi = 0,$$

when the substitution $r=1/z$ is made. The two independent solutions are:

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[2] For positive kinetic energy the essential singularity is given by (36).

(46) $\psi_1 = e^{-1/2z}$, and $\psi_2 = e^{1/2z}$.

Their ratio,

(47) $$f(z) = \frac{\psi_2}{\psi_1} = e^{1/z},$$

has an essential singularity at z=0 (r=∞). It can be considered as a limit of a rational function which is the ratio of a pole of order n at z=0, and a zero of order n at z=-1/n [3]. The ratio,

(48) $$\lim_{n\to\infty} \frac{(z+\frac{1}{n})^n}{z^n} = \lim_{n\to\infty}\left(1+\frac{1}{zn}\right)^n = e^{1/z},$$

has a finite limit coinciding with a transcendental function.

This occurs on the principal circle, or the positive axis of the half-plane.[3] The essential singularity thus consists of the merger of a pole of infinite order at z=0 and a zero of infinite order at r=0-.This permits us to interpret poles and zeros as opposite charges [3].

Since equation (43) has two singular points, one at the origin and the other at infinity, there are no limit points of the group of motions that separate the plane [21]. By transforming the singular point at infinity into an essential singularity, where an infinite number of poles will cluster, we introduce a boundary, either a principal circle or the real axis, depending if the domain is the disc or the half-plane, respectively. The transform involves introducing the kinetic energy which is presented by the last term in equation (43). The essential singularity has a dipole moment, which is related to a bound state, such as in the Schrödinger equation for the hydrogen atom, (43), in contrast to an unbound state as in Bessel's equation, (32), which has an infinite number of moments.

Let us look for a solution to (43) of the Fuchsian type, $\psi(r) = r^{l+1}\varphi(r)$. Then φ will be the solution of

(49) $$\varphi'' + 2\frac{l+1}{r}\varphi' + \left(\frac{\gamma}{r} - ¼\right)\varphi = 0.$$

Introducing the Euler operator, $\mathcal{D} = r\frac{d}{dr}$ [22], (49) can be reduced to the Fuchsian form:

(50) $$\mathcal{D}(\mathcal{D} + \boldsymbol{\lambda})\boldsymbol{\varphi} = -\boldsymbol{r}(\boldsymbol{\gamma} - ¼\boldsymbol{r})\boldsymbol{\varphi}.$$

The resonances, or roots of the left-hand side of the equation, are 0 and $-\boldsymbol{\lambda}$. This confirms that for small r the solution should behave as $r^{-\lambda}$. The stable manifold is parameterized by γ, the coefficient of the attractive coulombian potential.

Solving (50) recursively, we get the power expansion:

$$\varphi = r^{-\lambda}\{1 + \frac{\gamma}{\lambda-1}r + \frac{1}{2(\lambda-2)}\left(\frac{\gamma^2}{\lambda-1} - ¼\right)r^2 + \cdots\},$$

or, in terms of our original wave function,

(51) $$\psi = r^{-\lambda}\{1 + \frac{\gamma}{2l}r + \frac{1}{(2l-1)}\left(\frac{\gamma^2}{2l} - ¼\right)r^2 + \cdots\}.$$

The idea of such a power series solution is the same as Frobenius' 'trick' to consider logarithms as limiting cases of powers. Logarithmic solutions are admissible, and occur when the roots of the indicial equation coalesce. Equation (51) shows that it is an analytic function which has a branch pole of order $-l$ at r=0.

When we apply the same procedure to the fixed point at infinity by setting r=1/z, we get

(52) $$\mathcal{D}(\mathcal{D} - \boldsymbol{\lambda})\boldsymbol{\varphi} = -\frac{1}{z}\left(\boldsymbol{\gamma} - \frac{1}{4z}\right)\boldsymbol{\varphi},$$

---

[3] Points at infinity can be transformed to the principal circle by the linear-fractional transformation,

$$U(z) = \frac{iz+1}{z+i}.$$

which is not an equation of the Fuchsian type. At vanishing kinetic energy, (49) can be reduced to a Fuchsian type of differential equation by a transcendental change of variables, $R = e^{-1/z}$. Then introducing two radial coordinates, $R_0=R$ and $R_1=R\ln R$[22], (52) can be brought into the form:

(53) $$\mathcal{D}(\mathcal{D}+\boldsymbol{\lambda})\boldsymbol{\varphi} = \boldsymbol{\gamma}\,\frac{R_1}{R_0}\boldsymbol{\varphi},$$

where the 2-space scale operator is $\mathcal{D} = R^1\partial/\partial R_0$.

There is an analogy between the essential singularity at infinity of differential equations, like (32) and (18), and the limit point of a group, which is also an essential singularity [20]. The essential singularities of the group are the essential singularities of the automorphic function. The limit points either lie along the real axis in the half-plane, or on the principal circle. When an automorphic function is subjected to linear-fractional substitutions of the group, they will fill the half-plane or principal circle with fundamental regions that do not overlap and have no lacunae. However, in the immediate vicinity of a limit point, the automorphic function assumes any number of different values. The fundamental regions tend to cluster in infinite number about points on the principal circle, or on the real axis of the complex plane. Thus, *the behavior of an automorphic function at a limit point on the boundary is analogous to the confluence of two poles in a differential equation to produce an essential singularity at infinity.*

7. ANGULAR DEFECT AND GEOMETRIC PHASE

It appears that the phasor (5) violates the exterior angle theorem of what was known as the 16th proposition in Euclid's *Elements*. It states that an exterior angle of a triangle is greater than either remote interior angle, and so (5) would be negative. However, this is not true for spherical geometry in which the extension of the line from the remote vertex of the triangle meets the line parallel to the triangle is greater than a semi-circle [23]. Hence, all of what we have said previously applies to spherical geometry.

In contrast, geometric phase is well-known in hyperbolic geometry. There it is the angle defect that plays the role of the geometric phase. A classic example is the Wigner angle, which is the angle that two non-planar Lorentz boosts get rotated through [24].We might try to define the phasor as:

(54) $$\delta = \alpha - \beta,$$

for α>β. However, this is just the condition that the triangle is Euclidean, $\pi - \alpha + \beta + \delta = \pi$, as we found in equation (10). Rather, it is the defect.

(55) $$\Omega = \pi - (\pi - \alpha + \beta + \delta) > 0,$$

of the hyperbolic triangle, shown in Fig. 4, that is equal to its area. The defect of the hyperbolic triangle was associated

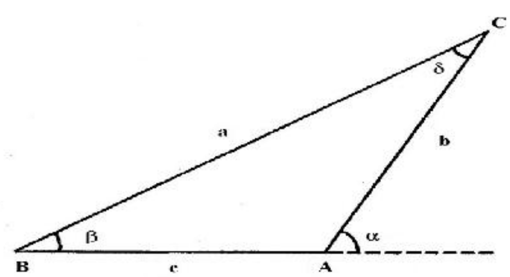

Fig. 4 The defect of the hyperbolic triangle is equal to its area. The first Lorentz boost is along the axis *BA*, while the second one occurs at an angle α along *AC*. The resultant boost is BC, and α goes from a minimum of 0 to a maximum of π/2.

a geometric phase in ref. [25], and references cited therein. The triangle in Fig. 4 represents the sum of two non-collinear Lorentz transformations or 'boosts.' The first is along the *BA* axis while the second is along *AC* making an angle α with respect to the first. It is suggestive that the logarithm of the ratio of the sine of (α-½Ω) to the sine of ½Ω is equal to the sum of the Euclidean measures of hyperbolicarc lengths:

(56) $ln\frac{\sin(\alpha-½\Omega)}{\sin(½\Omega)} = ½ln\frac{\cosh b+1}{coshb-1} + ½ln\frac{\cosh c+1}{coshc-1}$.

Whereas the hyperbolic tangent is the Euclidean measure of length of a 'straight' line in hyperbolic geometry, the hyperbolic cosine is the Euclidean measure of arc length [26].[4] For a right hyperbolic triangle α=½π, and the left-hand side of (54) becomes

(57) $$h = \ln\cot(½\Omega) = ½\ln\frac{1+\cos\Omega}{1-\cos\Omega},$$

which identifies Ω as the angle of parallelism. Two lines through a given point are parallel to a given line that make an angle Ω with respect to the perpendicular from this point to the given line whose distance is *h*. Ω is a function only of the length *h* of this perpendicular, and as the latter decreases, the former increases until it becomes a right angle when h becomes zero.

Combining (56) and (57) results in

(58) $$\tanh^{-1}\cos\Omega = \tanh^{-1}\cosh b\tanh^{-1}\cosh c = \tanh^{-1}\left\{\frac{\cosh b+\cosh c}{1+\cos a}\right\},$$

where the hyperbolic Pythagorean theorem, $\cosh a = \cosh b \cdot \cosh c$, has been used. Equation (58) is the well-known relation

(59) $$\cos\Omega = \frac{\cosh b+\cosh c}{1+\cosh a},$$

for the angle defect (55) [25].

8. CONCLUSIONS

Geometric phase is related to the curvature of non-Euclidean geometries. It has long been known that in hyperbolic geometry, two Lorentz transformations along different directions is accompanied by rotation whose angle is the defect of a hyperbolic triangle. Much less is known about geometric phase in spherical geometry, where the geometric phase arises from the multivaluedness of solutions to a second-order Fuchsian equation. The geometric phase is now related to the angular excess of a spherical triangle in the case of three regular singular points, or a lune in the case of two singular points. The quotient of any two solutions to the Fuchsian differential equation are functions automorphic with respect to a group of linear-fractional transformations that tessellate the half-plane, or unit disc, by the curvilinear triangles or lunes without lacunae and without overlap. The simple poles are conformally mapped onto the vertices of the fundamental regions where the automorphic function can only take any value just once. This imposes restrictions on the angular momentum quantum numbers which can no longer be integral, for, otherwise, the phase factors would become unity, which in the case of the Aharonov-Bohm effect would mean that the shift in the diffraction pattern disappears. Other spherical geometric examples are the phasor, the Pancharatnam phase of beams of interacting polarized light, and the Dirac monopole. The Dirac monopole is associated with the singularity of the Schrödinger equation in the limit of vanishing kinetic energy where it becomes a Fuchsian differential equation.

Fuchsian differential equations can be looked as limits of vanishing kinetic energy of the equations of mathematical physics where the essential singularities, prohibiting a blow up of the solution at infinity, are replaced by regular singular points. In the region of angular momentum quantum numbers where the angular motion represents 'centripetal' attraction, instead of repulsion, the geometric phase is one-half the area of a lune, which disappears when the pole at infinity becomes an essential singular thereby recovering the Schrödinger equation.

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[4] In spherical geometry, the spherical distance would be $cos^{-1}\,x = icosh^{-1}$(x), since $cosh^{-1}\,x$ is the hyperbolic measure of arc length.

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