Original Research Article

Computational Solution to Quantum Foundational Problems

Abstract

This paper argues that the requirement of applicableness of quantum linearity to any physical level from molecules and atoms to the level of macroscopic extensional world, which leads to a main foundational problem in quantum theory referred to as the "measurement problem", actually has a computational character: It implies that there is a generic efficient algorithm, which guarantees exact solutions to the Schrödinger equation for every physical system regardless of how many constituent microscopic particles it comprises. From the point of view of computational complexity theory, this requirement is equivalent to the assumption that the computational complexity classes **P** and **NP** are equal, which is considered very unlikely, since it would have some spectacular consequences for the natural sciences. As demonstrated in the paper, in the case of the opposite assumption that $\mathbf{P} \neq \mathbf{NP}$, the separation between a microscopic quantum system and a macroscopic apparatus (usually called the Heisenberg cut) would be justified as it would be impossible to overlap deterministic quantum and classical descriptions in order to obtain a rigorous derivation of complete properties of macroscopic objects from their microstates.

Keywords: Schrödinger equation · Quantum linearity · Reduction postulate · Born rule · Computational complexity · **P** versus **NP** question

1 Introduction

In quantum theory, the state of a microscopic particle is described by a state-vector $|\psi(t)\rangle$ (identified with a ray in the Hilbert space of the particle), whose subsequent time evolution is governed by the Schrödinger equation: $i\hbar\,\partial|\psi(t)\rangle/\partial t = H_p|\psi(t)\rangle$, where H_p is the Hamiltonian of the particle. According to the Copenhagen interpretation of quantum mechanics, the Schrödinger equation applies when the microscopic particle is evolving in isolation. But if the particle interacts with a macroscopic system or a measuring apparatus (like, for example, in the Stern–Gerlach setup), the reduction postulate and the Born rule must be used.

But then again, it seems unreasonable to have two incompatible dynamical laws governing the time evolution of the same particle: the deterministic Schrödinger equation for the events at the microscopic level and the stochastic reduction postulate for the events associated with micro–macro interactions. More logical and – in accordance with Occam's razor [1] – more intellectually economical is to believe that the Schrödinger equation is applicable to the whole class of physical entities

and hence governs all phenomena at both the microscopic and macroscopic levels (in fact, this is a belief accepted at the present by the great majority of practicing physicists).

Consequently, the time evolution of the state-vector $|\Psi(t)\rangle$ describing the state of the macroscopic composite system comprised of the microscopic particle and the apparatus may also be defined by the Schrödinger equation $i\hbar\partial|\Psi(t)\rangle/\partial t = H|\Psi(t)\rangle$, so that knowing the state of the composite system at an initial time $t = T_i$, one can in principle solve the Schrödinger equation with this initial condition to predict the state of the system at any future time $t = T_f$. As the Schrödinger Hamiltonian H is Hermitian, the predicted state at the time T_f is related to that at the initial time T_i by the deterministic relation $|\Psi(T_f)\rangle = U(T_f, T_i)|\Psi(T_i)\rangle$, where the transition operator $U(T_f, T_i)$ is unitary and completely specified by the Hamiltonian H of the composite system: $U(T_f, T_i) = \exp(-i(T_f - T_i)H/\hbar)$.

However, as soon as the prediction $|\Psi(T_f)\rangle$ is extracted from the Schrödinger equation, one will get – in virtue of linearity of this equation – a superposition state of the particle plus the apparatus (the infamous Schrödinger cat state), which is never experienced in our classical world. This paradox is especially puzzling since apparently the Schrödinger equation contains nothing prohibiting its application to macroscopic objects. Particularly, this equation does not say how large objects must be, before they can be said to obey the equations of Newtonian mechanics. Thence, a belief in the generic applicableness of the Schrödinger equation underlies a main foundational problem in quantum theory (known by the different names such as *macro-objectification problem*, *problem of definite outcomes*, *quantum measurement problem*, *problem of the emergence of classicality from quantum systems*, and so on).

A related problem also stemming from the belief in the generic applicableness of the Schrödinger equation is the origin of the probabilities in quantum theory. Namely, how can the probabilities come out of a deterministic, continuous and unitary time evolution given by the Schrödinger equation?

Those foundational problems have been debated for over eighty years, and during all these years, most of effort went into trying to resolve the problems within a physical theory (or physical theories). Thus, various attempts were made to change the interpretational rules of quantum theory as well as to modify its foundations (including attempts to use the principles of general relativity to change the formalism of quantum mechanics [2]). But since none of these attempts has gained general acceptance it might be time to look beyond physics and try to resolve those quantum foundational problems within a mathematical theory, specifically, computational complexity theory.

Indeed, along with the comprehensible and explicit assumption of the universality of the physical laws, the belief in the generic applicableness of the Schrödinger equation contains an additional, implicit assumption that has rather an *algorithmic* (or computational) character. This additional assumption is that there is a way to *efficiently* extract (i.e., extract *in a reasonable amount of time*) predictions about future states of physical systems – microscopic and macroscopic alike – from the Schrödinger equation. In other words, the hypothesis that the Schrödinger equation is applicable to everything from microscopic particles to macroscopic objects to the whole universe implies that there is a generic algorithm, which guarantees the efficient solution to the Schrödinger equation for every physical system, no matter how complicated and huge it is (i.e., how many constituent microscopic particles it comprises). But what if such a generic efficient algorithm does not really

exist? If it were so, then quantum theoretical constructions like "a quantum state of a macroscopic object" or "the wave function of the universe" would be nothing more than nontestable empty abstractions. Undeniably, this would have a crucial implication for the resolution of the quantum foundational problems.

The aim of this paper is to critically evaluate the assumption of the across-the-board efficient solvability of the Schrödinger equation in order to judge how plausible – from the point of view of computational complexity theory – this assumption is. The paper is structured as follows. First, considering the Schrödinger equation as a computational problem, it will be proved that this problem is **NP**-hard, which means that given an algorithm that solves Schrödinger's equation for all Hamiltonians, one would be able to solve all problems in the **NP** complexity class (covering most natural computer science problems). Next, it will be demonstrated that unless the class **NP** were found to be equal to the class **P** containing computational problems that are said to be easy or tractable, Schrödinger's equation would be merely unsolvable for macroscopic systems and accordingly inapplicable to their time evolution portrayal. Finally, it will be shown that randomness is entered in pure Hamiltonian evolution as a way to obtain the prediction about the state of a microscopic system interacting with its macroscopic environment without solving the environmental Schrödinger equation – an intractable computational problem.

2 Schrödinger's equation as a computational problem

Let us start our evaluation by defining the Schrödinger equation as a computational problem, which we will call the problem Φ_Ψ:

> *Given the Schrödinger Hamiltonian H, what is the solution $|\psi(t)\rangle$ to the Schrödinger equation $i\hbar\,\partial|\psi(t)\rangle/\partial t = H|\psi(t)\rangle$?*

Despite the fact that in this form the differential operator $\partial/\partial t$ and the Hamiltonian H are just abstract operators acting on kets $|\psi(t)\rangle$, abstract objects, the problem Φ_Ψ may be understood as *a functional computational problem* once this form is projected into the position basis $\{|\mathbf{r}\rangle\}$: $i\hbar\,\partial\Psi(\mathbf{r},t)/\partial t = H\Psi(\mathbf{r},t)$, where the wave function $\Psi(\mathbf{r},t)$ is the scalar product $\langle\mathbf{r}\,|\psi(t)\rangle$, $\mathbf{r} = (\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$ denotes the sets of position vectors, and N is the system constituent particle number. In this way, to solve an instance of the functional problem Φ_Ψ would mean to set up the Hamiltonian H for a particular system accounting for the kinetic and potential energy of all the particles constituting the system and having inserted H into the Schrödinger equation to solve the resulting partial differential (in general, time-dependent) equation for the state of the system at time t represented by the vector $|\psi(t)\rangle$, which stands for the wave function $\Psi(\mathbf{r},t)$.

As every function computational problem can be turned into *a decision computational problem* (i.e., a question with only 'yes'-or-'no' answer), we can easily change the functional problem Φ_Ψ into the decision problem Π_Ψ by incorporating a set of additional restraints R imposed on the solution $|\psi(t)\rangle$ (or its eigenvalue) or/and on a set of polynomially bounded functions of the solution $|\psi(t)\rangle$ (or its eigenvalue) into the problem Φ_Ψ:

> *Given the Schrödinger Hamiltonian H and a set of restraints R, is there a solution $|\psi(t)\rangle$ to the Schrödinger equation $i\hbar\,\partial|\psi(t)\rangle/\partial t = H|\psi(t)\rangle$ subject to those restrains R?*

The decision problem Π_Ψ can equally be defined as the set S_Ψ of inputs – various Hamiltonians H and corresponding restraints R – for which Π_Ψ returns 1 (i.e. 'yes'):

$$S_\Psi = \left\{ H,\ R: \quad \Pi_\Psi\left(i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H|\psi(t)\rangle \ \wedge \ R \right) = 1 \right\} \quad . \tag{1}$$

Let $A(\Phi_\Psi)$ denote a generic algorithm that solves the functional problem Φ_Ψ on all instances, i.e., it solves the Schrödinger equation for all Hamiltonians H. Obviously, such an algorithm can easily be modified to solve all instances of the decision problem Π_Ψ as well: the algorithm $A(\Pi_\Psi)$ for solving Π_Ψ will first call the algorithm $A(\Phi_\Psi)$ as a subroutine to solve the Schrödinger equation for the state $|\psi(t)\rangle$ and then decide in polynomial number of steps whether the ensuing state $|\psi(t)\rangle$ (or its eigenvalue) satisfies the restraints R.

Suppose the vector $|\psi(t)\rangle$ is the solution to the Schrödinger equation for the given Hamiltonian H and subject to the given restrains R. Let us show that the decision problem Π_Ψ can be quickly verified, i.e., one can check whether Π_Ψ returns 1 for these H and R in reasonable time. To accomplish this, one should substitute the solution $|\psi(t)\rangle$ back into the expression for Π_Ψ and estimate the runtime complexity of the operations needed to prove that Π_Ψ indeed returns 1.

Let L be the minimal number of elementary operations sufficient to compute the effects of the differential operator $\partial/\partial t$ and the Hamiltonian H on the known vector $|\psi(t)\rangle$; we will call L the complexity of verification. In the position basis $\{|\mathbf{r}\rangle\}$ the complexity L can be presented as follows:

$$L\left(\frac{\partial |\psi(t)\rangle}{\partial t},\ H|\psi(t)\rangle \right) = L\left(\frac{\partial \Psi}{\partial t}, \frac{\partial^2 \Psi}{\partial \mathbf{r}_1^2}, \ldots, \frac{\partial^2 \Psi}{\partial \mathbf{r}_N^2}, \frac{\partial \Psi}{\partial \mathbf{r}_1}, \ldots, \frac{\partial \Psi}{\partial \mathbf{r}_N}, \Psi \right) \quad . \tag{2}$$

In (2) only nonscalar multiplications/divisions are considered contributed to the complexity of verification, whereas additions/subtractions and multiplications by arbitrary scalars $k(\mathbf{r}, t)$ are allowed for free. The easiest method of approximating partial derivative $\partial F/\partial q_j$, which uses the finite difference quotient

$$\frac{\partial F}{\partial q_j} = \frac{F\left(q_1, \ldots, q_j + h_j, \ldots, q_n\right) - F\left(q_1, \ldots, q_j, \ldots, q_n\right)}{h_j} \tag{3}$$

with $h_j \to 0$, will lead to the following upper bound of the verification complexity

$$L\left(\frac{\partial |\psi(t)\rangle}{\partial t},\ H|\psi(t)\rangle \right) \leq O\left(N^2\right) \cdot cost(\Psi(\mathbf{r}, t)) \quad , \tag{4}$$

where $cost(\Psi(\mathbf{r}, t))$ denotes the computational cost of the wave function evaluation at particular numerical values $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ and t. For the interpretation of the wave function to make sense, it must be feasible to evaluate the function $\Psi(\mathbf{r}, t)$; otherwise, it would be impossible to use $\Psi(\mathbf{r}, t)$ to compute a measurable observable of the quantum system. According to Cobham's thesis [3], the function $\Psi(\mathbf{r}, t)$ can be feasibly evaluated on some computational device only if $\Psi(\mathbf{r}, t)$ can be evaluated in polynomial time. Correspondingly, one gets that the verification complexity L must be upper bounded by a polynomial, which means that the decision problem Π_Ψ is in the

NP complexity class of computational problems, whose solutions can be verified in polynomial time.

Suppose that it is possible to encode a specific instance of a given **NP** problem Π_C in a certain Hamiltonian. More precisely, suppose that the ground state of the Hamiltonian $H_C(Z)$ encodes the solution to an instance of 3-SAT [4], a **NP**-complete problem – i.e., a problem in **NP** that all other problems in **NP** can be converted to by a transformation of inputs in polynomial time. Formally, the problem Π_C can be defined as the set S_C of inputs Z, which satisfy all of the Boolean clauses that specify the instance of 3-SAT, or, in other words, for which the problem Π_C returns 'yes':

$$S_C = \left\{ \, Z : \quad \Pi_C(Z) = 1 \, \right\} \quad . \tag{5}$$

Let an adiabatically evolving system be characterized by the Hamiltonian $H(t)$, which is slowly varying, and so at any instant of time t this system remains in the state $|\psi(t)\rangle$ close to the instantaneous ground state of the Hamiltonian $H(t)$. Suppose that at the initial time T_i the ground state $|\psi(T_i)\rangle$ of the Hamiltonian $H(t)$ is known, but at time $T_f = T_i + \tau$ the Hamiltonian $H(T_f)$ coincides with another Hamiltonian $H_C(Z)$, whose ground state $|\psi_0(T_f)\rangle$ encodes the solution to the given problem Π_C in such a way that the problem Π_C returns 'yes' when the ground state $|\psi_0(T_f)\rangle$ corresponds to the instantaneous eigenvalue $E_{T_f} = 0$. Then, the given problem Π_C and the problem $\Pi_{\Psi_{\mathrm{C}}}$

$$S_{\Psi_{\mathrm{C}}} = \left\{ \, H(t), E_{T_f} = 0 : \quad \Pi_{\Psi_{\mathrm{C}}} \left(i\hbar \, \frac{\partial}{\partial t} \, |\psi(t)\rangle = H(t) |\psi(t)\rangle \ \wedge \ H(T_f) \, |\psi(T_f)\rangle = 0 \right) = 1 \, \right\} \quad , \tag{6}$$

which is a particular instance of the decision problem Π_Ψ, where the restraint R: $E_{T_f} = 0$ is imposed on the instantaneous eigenvalue E_{T_f} of the Hamiltonian $H(t)$ specified as

$$\forall t \in [T_i, \ T_i + \tau] : \qquad H(t) \cong \left(1 - \frac{t - T_i}{\tau} \right) H(T_i) + \frac{t - T_i}{\tau} H_C(Z) \quad , \tag{7}$$

will be *equisatisfiable*, that is, Π_C will return 'yes' whenever $\Pi_{\Psi_{\mathrm{C}}}$ will, and vice versa. Hence, if the Hamiltonians $H(T_i)$ and $H_C(Z)$ were easy to construct, i.e., they could be composed (described) in a polynomial amount of steps, then the algorithm $A(\Pi_\Psi)$ for solving the decision problem Π_Ψ would solve the given problem Π_C with only polynomially more work. And as Π_C is a **NP**-complete problem, so Π_Ψ would be as well. In consequence, all other problems of the **NP** class would be convertible to Π_Ψ in polynomial time, and thus the algorithm $A(\Pi_\Psi)$ would be able to solve all other problems in the **NP** complexity class.

As demonstrated in the papers [5, 6], which studied instances of Exact Cover and the satisfiability problem – both **NP**-complete problems, such Hamiltonians $H(T_i)$ and $H_C(Z)$ are really straightforward to construct. This implies that the computational problem Π_Ψ is **NP**-complete, and so the algorithm $A(\Phi_\Psi)$ for solving the functional problem Φ_Ψ can be quickly modified for solving any **NP** problem. In other words, given the algorithm $A(\Phi_\Psi)$ that solves the Schrödinger equation for all Hamiltonians, any problem in the **NP** class can be solved with only polynomially more work.

This means that *the problem Φ_Ψ of solving the Schrödinger equation is **NP**-hard*, i.e., at least as hard as (or harder than) any problem in the **NP** class. A consequence of the **NP**-hardness of the Schrödinger equation is that if the algorithm $A(\Phi_\Psi)$ were efficient, the class **NP** would be equal to the class **P** of computational problems solvable in polynomial time.

Although whether **P** is equal to **NP** or not is a major unresolved problem in computer science, the assumption of the across-the-board efficient solvability of the Schrödinger equation embraces the attitude that **P**=**NP**, contrary to widespread (among computational theory scientists) belief that **P**≠**NP** [7].

3 How Schrödinger's cat is explained by computational complexity theory

Due to its dimensions and constant interaction with the environment, a macroscopic system contains a huge and essentially unchecked number of constituent microscopic particles. Following this argument, one can remark that to solve exactly the Schrödinger equation for a macroscopic system in a reasonable amount of time would only be possible if the generic efficient algorithm having the ability to exactly solve the Schrödinger equation for *an arbitrary system* were to exist (where arbitrary means "any" and is here synonymous with *worst-case*). Only in this case (in which the **NP**-hard problem Φ_Ψ would turn out to be tractable and thus **P** and **NP** would collapse into one class), the accurate number of the macroscopic system constituent particles would be practically irrelevant to the time needed to exactly solve the Schrödinger equation since no amount of extra particles added to the system would be able to significantly slowdown the performance of the efficient algorithm.

But if the generic efficient algorithm did not exist (i.e., if **P**≠**NP** and so the problem Φ_Ψ were intractable), then only one other algorithm guaranteeing to find a solution (if it exists) to the Schrödinger equation for any system would be *brute-force search* (or exhaustive search) consisting of exhaustive examination of all possibilities, i.e., all possible candidates for the solution. However, the number of elementary operations required by a brute-force algorithm to reach the exact solution to the Schrödinger equation for a given system is proportional to the dimensionality of the system state space, which (according to the postulates of quantum mechanics) scales exponentially with N, the number of the system constituent particles. As a result, the number of elementary operations needed to exactly solve the Schrödinger equation for a macroscopic system would be of the same magnitude as an exponential 2^{N_A} of the Avogadro's number $N_\mathrm{A} \sim 10^{24}$.

Let us stipulate, for example, that for non-real-time applications a running time of one year is reasonable. Then, to find the solution to the Schrödinger equation for a macroscopic system by exhaustive search in one year will require a computer to execute each elementary operation on the order of

$$\frac{1 \text{ year}}{2^{10^{24}}} \approx \frac{3 \cdot 10^7 \text{ seconds}}{10^{3 \cdot 10^{23}}} \sim 10^{-3 \cdot 10^{23}} \text{ seconds} \quad , \tag{8}$$

which is by a considerable margin less than one Planck time roughly equal to 10^{-43} seconds. But within the framework of the laws of physics, for times less than one Planck time apart one can neither measure nor detect any change. Hence, even if there was a device able to do an elementary computing operation in a time as short as ratio (8), one would be able to neither measure the result of this operation nor simply detect that this device would have done something. Moreover, for ratio (8) there would not be much of a difference between running times of one year and, say, of one hundred billion years ($\sim 10^{18}$ seconds). So, unless the laws of physics (as we understand them today) were wrong (and consequently some physical processes of a uniquely new nature were possible that allowed construction of übercomputers – a sort of extraordinary, superior computing devices), no computer would ever be able to execute 2^{N_A} operations in any reasonable amount time.

This implies that in the case, in which the problem Φ_Ψ would be intractable, the deterministic quantum model of a macroscopic system (built around the exact solutions to the system Schrödinger equation) would be without predictive content inasmuch as there would be no practical means to extract the prediction about the system future state from the Schrödinger equation. In this manner, a Schrödinger cat state – as a linear combination of the exact (and orthogonalized) solutions to the system Schrödinger equation – would be predictively contentless and for this reason unavailable for inspection.

By contrast, a stochastic quantum model of a macroscopic system that is built around inexact (i.e. with a degree of uncertainty) solutions to the system Schrödinger equation might have predictive content even if the problem Φ_Ψ was intractable.

In fact, a brute-force algorithm can reach the solution to the Schrödinger equation in reasonable time but only if the state space of a system is limited (as in the case of a system consisting of a few microscopic particles completely isolated from the environment) or when there are system-specific heuristics that can be used to reduce the set of all possible candidate solutions to a limited size.

Take, for example, a composite system comprised of two interacting systems – a microscopic particle ("test-particle"), whose states are controlled or measured, and a macroscopic system ("environment"), whose constituent microscopic particle states are uncontrolled and unmeasured. As the environment microscopic states are ignored, the set of all possible candidate solutions to the Schrödinger equation for the given composite system can *effectively* (i.e., for all practical purposes) be reduced to the set of the test-particle candidate solutions. In doing so, one would get an inexact yet fast (and so feasible) solution describing (in probability terms) the test-particle interacting with the environment.

Let us show in detail how this heuristics works. At the time $T_f = T_i + \tau, \tau \geq 0$ the state of the composite system "particle + environment" (whose Hilbert space is the tensor product $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_\varepsilon$ of the two Hilbert spaces – $\mathcal{H}_p$ of the particle and $\mathcal{H}_\varepsilon$ of the environment) is related to that at the initial time T_i by the following deterministic relation:

$$|\Psi_{T_i+\tau}\rangle = U(T_i+\tau, T_i)\,|\Psi_{T_i}\rangle = \exp\left(-\frac{i\tau}{\hbar}H\right)|\Psi_{T_i}\rangle \quad . \tag{9}$$

In this relation, the initial state-vector of the composite system $|\Psi_{T_i}\rangle$ is the direct product of the

state-vector $|\psi_{T_i}\rangle$ in $\mathcal{H}_p$ and the state-vector $|\varepsilon_{T_i}\rangle$ in $\mathcal{H}_\varepsilon$

$$|\Psi_{T_i}\rangle = |\psi_{T_i}\rangle \otimes |\varepsilon_{T_i}\rangle = \sum_j c_j \, |\psi_j\rangle \otimes \sum_k \alpha_k \, |\varepsilon_k\rangle \quad , \tag{10}$$

where the orthonormal basis vectors $|\psi_j\rangle$ and $|\varepsilon_k\rangle$ spanning the spaces $\mathcal{H}_p$ and $\mathcal{H}_\varepsilon$ are the exact solutions to the Schrödinger equations for the particle Hamiltonian H_p

$$i\hbar \, \frac{\partial}{\partial t} \, |\psi_j\rangle = H_p |\psi_j\rangle \tag{11}$$

and for the environment Hamiltonian H_ε

$$i\hbar \, \frac{\partial}{\partial t} \, |\varepsilon_k\rangle = H_\varepsilon |\varepsilon_k\rangle \quad , \tag{12}$$

c_j and α_k denote complex coefficients of the superpositions, while the Hamiltonian of the composite system H can be presented (at least during interaction time τ) entirely by the interaction term H_{int}

$$H \cong H_{\text{int}} = \sum_j |\psi_j\rangle\langle\psi_j| \otimes \sum_k A_{jk} \, |\varepsilon_k\rangle\langle\varepsilon_k| \quad , \tag{13}$$

in which $|\psi_j\rangle\langle\psi_j|$ and $|\varepsilon_k\rangle\langle\varepsilon_k|$ are the operators acting on $\mathcal{H}_p$ and $\mathcal{H}_\varepsilon$, respectively, and A_{jk} stand for the interaction coefficients. So, as it is readily seen from the following expression

$$|\Psi_{T_i+\tau}\rangle = \left(I - \frac{i\tau}{\hbar} \sum_j |\psi_j\rangle\langle\psi_j| \otimes \sum_k A_{jk} \, |\varepsilon_k\rangle\langle\varepsilon_k| \right) \sum_j c_j \, |\psi_j\rangle \otimes \sum_k \alpha_k \, |\varepsilon_k\rangle \quad , \tag{14}$$

to extract information about the state of the composite system $|\Psi_{T_i+\tau}\rangle$ at some moment $T_i + \tau$ one has to know the basis vectors $|\psi_j\rangle$ and $|\varepsilon_k\rangle$, but to obtain them the Schrödinger equations (11) and (12) must be solved, obviously.

The equation (11) can assuredly be solved by brute-force search in reasonable time (due to the limited dimensionality of the test-particle state space $\mathcal{H}_p$, which before the interaction may be considered as completely isolated from the environment state space $\mathcal{H}_\varepsilon$), whereas the equation (12) cannot. Therefore, to obtain information about the state $|\Psi_{T_i+\tau}\rangle$ in the case, in which $\mathbf{P} \neq \mathbf{NP}$ and hence brute-force search cannot be avoided, we will allow uncertainties in the interaction coefficients A_{jk} associated with different microscopic configurations of the environment – arrangements of its microscopic constituent particles (in view of the fact that those particles are uncontrolled and unmeasured) such that

$$\forall j, k: \qquad A_{jk} = \widetilde{A_j} + a_{jk}(\omega) \quad , \tag{15}$$

where $\widetilde{A_j}$ are estimates for the interaction coefficients, which can be taken as roughly proportional to the number of electrons in the environment (given that the interaction between the test-particle and its environment can be assumed to be due to the Coulomb force), and $a_{jk}(\omega)$ are real-valued random variables of equal (among different environment microstates) distribution

$$\forall j,k: \qquad a_{jk}(\omega) \sim a_j(\omega) \tag{16}$$

defined on a set of possible outcomes, the sample space Ω, as

$$\forall j: \qquad \left\{\omega \in \Omega: \quad |a_j(\omega)| \leq \widetilde{A_j}\right\} \quad . \tag{17}$$

Introduced in this manner uncertainties will effectively convert the operator $\sum_k A_{jk}\,|\varepsilon_k\rangle\langle\varepsilon_k|$ (which acts on the environment state space $\mathcal{H}_\varepsilon$) into the product of a stochastic scalar and the unit operator $\sum_k |\varepsilon_k\rangle\langle\varepsilon_k| = \hat{1}$

$$\forall j: \qquad \sum_k A_{jk}\,|\varepsilon_k\rangle\langle\varepsilon_k| \sim \left(\widetilde{A_j} + a_j(\omega)\right)\hat{1} \quad . \tag{18}$$

In turn, the resulted equalities (18) will transform the deterministic expression (13) for the interaction Hamiltonian into a stochastic one

$$H_{\rm int}(\omega) \sim \sum_j \left(\widetilde{A_j} + a_j(\omega)\right)|\psi_j\rangle\langle\psi_j| \otimes \hat{1} \tag{19}$$

and in this way will preclude the necessity of solving the environmental Schrödinger equation (12) to obtain the incomplete (as the environmental microstates $|\varepsilon_{T_i}\rangle$ are unknown) prediction for the final state $|\Psi_{T_i+\tau}\rangle$

$$|\Psi_{T_i+\tau}\rangle = \left(I - \frac{i\tau}{\hbar}H_{\rm int}(\omega)\right)\sum_j c_j\,|\psi_j\rangle \otimes |\varepsilon_{T_i}\rangle \sim |\psi_{T_i+\tau}(\omega)\rangle \otimes |\varepsilon_{T_i}\rangle \quad , \tag{20}$$

which will, nonetheless, contain information – albeit inexact one – about the state of the test-particle at the final time $T_i + \tau$ after the interaction with the environment:

$$\left\{\omega \in \Omega: \quad |\psi_{T_i+\tau}(\omega)\rangle = \sum_j c_j\,|\psi_j\rangle \exp\left(-\frac{\widetilde{A_j} + a_j(\omega)}{\hbar}\,i\tau\right)\right\} \quad . \tag{21}$$

As follows, the random state-vector $|\psi_{T_i+\tau}(\omega)\rangle$ does not represent a single, fixed final state of the test-particle; rather it takes on a set of possible different final states. That is to say, the vector $|\psi_{T_i+\tau}(\omega)\rangle$ associates states of the test-particle at the final time $T_i + \tau$ with instances ω of a yet-to-be-performed experiment, so that $|\psi_{T_i+\tau}(\omega)\rangle$ will vary from instance to instance as the experiment is repeated.

This means that there must be a probability distribution associated with the random state-vector $|\psi_{T_i+\tau}(\omega)\rangle$ that allows the computation of the probabilities of the possible final states. But in accordance with the postulates of quantum mechanics, the state-vector of the test-particle $|\psi_{T_i+\tau}(\omega)\rangle$ determines everything that can be known about this test-particle. It can be inferred from here that the probability distribution associated with the state-vector $|\psi_{T_i+\tau}(\omega)\rangle$ must be determined by the vector $|\psi_{T_i+\tau}(\omega)\rangle$ itself (otherwise, quantum theory cannot be considered complete).

Yet, technically, using the complex vector $|\psi_{T_i+\tau}(\omega)\rangle$ is impossible to define a probability measure – a real-valued non-negative function that must return results in the unit interval $[0,1]$ (producing 0 for the empty set and 1 for the entire sample set Ω) and satisfy the countable additivity property. On the other hand, according to the Gleason's theorem [8, 9], if one would like to assign a probability measure to the vector $|\psi_{T_i+\tau}(\omega)\rangle$, the only possible choice is $|\langle\varphi|\,\psi_{T_i+\tau}(\omega)\rangle|^2$, the modulus squared of the scalar product of $|\psi_{T_i+\tau}(\omega)\rangle$ and some arbitrary but fixed vector $|\,\varphi\rangle$. Choosing the initial state-vector $|\psi_{T_i}\rangle$ of the test-particle as such a fixed vector, one will have the following probability measure turning the sample space Ω into a probability space:

$$\left\{\omega\in\Omega:\quad |\langle\psi_{T_i}|\,\psi_{T_i+\tau}(\omega)\rangle|^2\right\}=[0,1]\quad , \tag{22}$$

where

$$|\langle\psi_{T_i}|\,\psi_{T_i+\tau}(\omega)\rangle|^2=\left|\left(\sum_g c_g^*\,\langle\,\psi_g|\right)\left(\sum_j c_j\,|\psi_j\rangle\exp\left(-\frac{\widetilde{A_j}+a_j\,(\omega)}{\hbar}\,i\tau\right)\right)\right|^2\quad , \tag{23}$$

provided that $\sum_j |c_j|^2=1$. Performing the experiment many times, one can find a typical (of this experiment) value for the probability of transitioning the test-particle from the initial state $|\psi_{T_i}\rangle$ to the final state $|\psi_{T_i+\tau}\rangle$ by averaging $|\langle\psi_{T_i}|\,\psi_{T_i+\tau}(\omega)\rangle|^2$ over the entire sample set Ω:

$$P\left(|\psi_{T_i}\rangle\to|\psi_{T_i+\tau}\rangle\right)=\overline{|\langle\psi_{T_i}|\,\psi_{T_i+\tau}(\omega)\rangle|^2}\quad . \tag{24}$$

Assuming that the state space of the test-particle is the n-dimensional complex Hilbert space C^n (where n is limited) and for the sake of simplicity supposing that all $|c_j|^2=\frac{1}{n}$, one can find from Eq.(23)

$$|\langle\psi_{T_i}|\,\psi_{T_i+\tau}(\omega)\rangle|^2=\frac{1}{n}+\frac{2}{n^2}\sum_{j=1}^{n-1}\sum_{g=j+1}^{n}\left(\cos\Xi_{jg}\,\cos\xi_{jg}(\omega)-\sin\Xi_{jg}\,\sin\xi_{jg}(\omega)\right)\quad , \tag{25}$$

where the angles Ξ_{jg} are

$$\Xi_{jg}=\frac{\widetilde{A_j}-\widetilde{A_g}}{\hbar}\,\tau\quad , \tag{26}$$

and the random angles $\xi_{jg}(\omega)$ are defined as

$$\xi_{jg}(\omega) = \frac{a_j(\omega) - a_g(\omega)}{\hbar}\,\tau \quad . \tag{27}$$

The total span of the random angles can be assessed by their maximum and minimum limits:

$$\max_{\omega\in\Omega}\xi_{jg}(\omega) = -\min_{\omega\in\Omega}\xi_{jg}(\omega) \approx \frac{\widetilde{A_j} + \widetilde{A_g}}{\hbar}\,\tau \quad ; \tag{28}$$

so, assuming that the random angles $\xi_{jg}(\omega)$ are spread uniformly within these limits (because the uniform distribution is the one that makes the least claim to being informed about the interaction coefficients A_{jk} associated with uncontrolled and unmeasured microscopic configurations of the environment beyond knowing the approximate limits of A_{jk}), one can find the average values of the functions of $\xi_{jg}(\omega)$ over the sample set Ω:

$$\overline{\cos\xi_{jg}(\omega)} \approx \frac{\hbar}{\left(\widetilde{A_j} + \widetilde{A_g}\right)\tau}\sin\frac{\widetilde{A_j} + \widetilde{A_g}}{\hbar}\,\tau \quad , \tag{29}$$

$$\overline{\sin\xi_{jg}(\omega)} = 0 \quad . \tag{30}$$

To define the number of electrons in the environment one can assume that only those environmental electrons that are within a radius of $R = cT$ (where c is the speed of light, T is the time allotted for the interaction, which is typically of order 10^{-3} seconds) can influence the test-particle. It is obvious that in the normal environment the number of electrons within the causal horizon R (and thus the estimates $\widetilde{A_j}$ and $\widetilde{A_g}$) will be of a considerable magnitude. This means that after a very short period of the interaction, the argument of the sine function in (29) will be close to infinity and consequently the whole right hand side of (29) will be close to zero. So, if the state of the test-particle is initially given by the superposition state $|\psi_{T_i}\rangle = \sum_j^n \frac{1}{\sqrt{n}}\,|\psi_j\rangle$, then at the time $T_i + \tau$ the transition probability (24) will be equal to that following from the Born rule

$$P\left(|\psi_{T_i}\rangle \to |\psi_{T_i+\tau}\rangle\right) \approx P\left(|\psi_{T_i}\rangle \to |\psi_j\rangle\right) = \frac{1}{n} \tag{31}$$

meaning that upon the interaction with the environment the initial state $|\psi_{T_i}\rangle$ will 'collapse' in the sense that $|\psi_{T_i}\rangle$ will change to $|\psi_j\rangle$.

In this way, the reduction postulate and the Born rule can be considered as a mere shortcut, a way to get the last result without using the presented above heuristic.

4 Final remarks

Since the earliest years of quantum theory, it has become increasingly evident that the rapid rate, at which the Schrödinger equation grows to be more complicated as the size of a system increases, makes the task of deriving complete properties of macroscopic objects from their microstates simply hopeless. Yet, such a problem was never considered as something fundamental since one may always hope that the Schrödinger equation will certainly be solved someday at least numerically, because numerical solutions are always reachable if only enough computational resources are thrown at them.

However, in all likelihood, such is not the actual state of things in our real physical world. As it was shown in this paper, solving the Schrödinger equation for any Schrödinger Hamiltonian is a problem at least as hard as the hardest problems in the **NP** computational complexity class. This implies that unless **P** and **NP** collapse into one class (which is very unlikely as it would imply many startling results that are currently believed to be false), coming up with the exact solution to Schrödinger's equation for an arbitrary system will inevitably involve exhaustive search over an exponentially large set of all possible candidate solutions. As a result, computational resources required by an algorithm using brute force will grow so rapidly with the system microscopic constituent particle number that bringing any additional resources to bear on the algorithm will be just of no value. And so, for anyone living in the real physical world (of limited computational recourses) the Schrödinger equation will turn out to be simply unsolvable for macroscopic objects and accordingly inapplicable to their time evolution portrayal.

In other words, in the case, in which **P** class is not equal to **NP**, it is impossible to overlap deterministic quantum and classical descriptions in order to obtain a rigorous derivation of classical properties from quantum mechanics.

As said, another foundational problem in quantum theory is how to reconcile the linear, deterministic evolution described by the Schrödinger equation with the occurrence of random, definite measurement outcomes. In this paper, randomness (and associated with it probability) is entered as a way to obtain the prediction about the final state of the test-particle interacting with the environment without solving the environmental Schrödinger equation – an intractable computational problem. For to allow statistical uncertainty in the description of a system is effectively equivalent to making the description less detailed, which in turn reduces the number of possible candidate solutions, needed to search over to find the correct one. Clearly, had the Schrödinger equation have the efficient generic algorithm that could solve it exactly for all Hamiltonians, the predicted state of the composite system "particle + environment" would be given by a deterministic expression deprived of any randomness.

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