A Review Of The Construction Of Particular Measures

Yao Elikem Ayekple, William Obeng-Denteh, Joshua Amevialor, Acquah Esther

Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana.

jegyaoo@gmail.com, obengdentehw@yahoo.com, jamkwo@gmail.com

Abstract

The first measure one usually comes into contact with in undergraduate mathematical studies is the Lebesgue measure and seeing how it is applied to the Lebesgue integral to extend considerably the Riemann integral, it doesn't take very much else to arouse one's interest in the study of measures and their construction with the hope/intent of eliciting their usefulness and how they are applied to other areas of mathematics. The Carathéodory extension theorem and the Carathéodory-Hahn theorem which are invoked subsequently in the construction of some measures are stated without proof. A large class of measures exist and this paper illustrates the construction of some of these measures including the Radon measure, the Hausdorff measure, the Lebesgue-Stieltjes measure, the Lebesgue measure in $\mathbb{R}^n$ and Product measures.

Keywords: σ-algebra, Premeasure, Carathéodory-Hahn theorem, Carathéodory extension theorem, Borel measure, Radon measure, Hausdorff measure, Lebesgue-Stieltjes measure, Lebesgue measure.

INTRODUCTION

Like most concepts in mathematics, measure has a formal rigorous definition, but according to wikipedia; a measure on a set can be thought of as a systematic way to assign a number to each suitable subset of that set, intuitively interpreted as its size. In this sense, a measure is a generalization of the concepts of length, area, and volume. A particularly important example is the Lebesgue measure on a Euclidean space, which assigns the conventional length, area, and volume of Euclidean geometry to suitable subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. For instance, the Lebesgue measure of the interval [0, 1] in $\mathbb{R}$ is its length, specifically 1.

Generally, a measure is a function that assigns a non-negative real number or $+\infty$ to (certain) subsets of a set $\mathbf{X}$. It must assign 0 to the empty set and be countably additive: the measure of a "large" subset that can be decomposed into a finite or countable number of "smaller" disjoint subsets, is the sum of the measures of the "smaller" subsets. In general, if one wants to associate a consistent size to each subset of a given set while satisfying the other axioms of a measure, one only finds trivial examples like the counting measure, which defines the measure of a set, S, simply as the number of elements in S. This problem was resolved by defining measure only on a sub-collection of all subsets, called measurable subsets, which are required to form a σ-algebra. This means that countable unions, countable intersections and complements of measurable subsets are measurable.

All ordinary concepts of length or area or volume apply only to reasonably regular sets. Modern measure theory is remarkably powerful in that an extraordinary variety of sets are regular enough to be measured; but there are still some limitations, and when studying any measure, a proper understanding of the class of sets which it measures is indispensable [4]. This leads us to the existence of non-measurable sets. Generally speaking, non-measurable sets are sets which fail to be measurable with respect to a translation-invariant measure.

Background

[3] gives an outline of the background of measure theory, which spans from the early notion of measure introduced by Italian Guiseppe Peano in 1883, to the Austrian Johann Radon's 1913 Habilitation work where he began the study of abstract measures and integrals more general than those of Lebesgue in $\mathbb{R}^n$. Generally, measure theory was developed in successive stages during the late 19th and early 20th centuries by Émile Borel, Henri Lebesgue, Johann Radon, Camille Jordan and Maurice Fréchet, among others.

The main applications of measures are in the foundations of the Lebesgue integral, in Andrey Kolmogorov's axiomatization of probability theory and in ergodic theory. The general theory of integration is a significant advancement from the Riemannian integration known from calculus. According to [3], although many applications in natural sciences triggered the development of general integration and measure theory, the theory of probability has become the primary client of abstract measure even prior to integration.

In integration theory, specifying a measure allows one to define integrals on spaces more general than subsets of the Euclidean space. Probability theory considers measures that assign to the whole set the size 1, and considers measurable subsets to be events whose probability is given by the measure. Ergodic theory considers measures that are invariant under, or arise naturally from a dynamical system [15]. Toward the end of the nineteenth century, it became clear to many mathematicians that the Riemann integral (about which one learns in calculus classes) should be replaced by some other type of integral, more general and more flexible, better suited for dealing with limit processes [11].

Notable among the shortcomings of the Riemann integral being the fact that the point-wise limit of a sequence of Riemann integrable functions is not necessarily Riemann integrable an example being the Dirichlet function over [0, 1]. An attempt to overcome these limitations resulted in the creation of the Lebesgue integral, which is based on the Lebesgue measure. Hence any set or function that is Lebesgue measurable is also Lebesgue integrable. Many sets and functions are measurable, for example, the Cantor set and the characteristic function. The Cantor set, though being a perfect, totally disconnected set, can have a continuous function defined on it, but the Riemann integral of such a function cannot be found. Also, the characteristic function is not continuous and also not Riemann integrable. Hence in the theory of Lebesgue integral, measurability replaces the need for complete continuity [2], [6], [8].

PREREQUISITES AND METHODOLOGY

We note the following definitions and theorems.

Definition: Let X be a set. A collection $\mathcal{M}$ of subsets of X is called a ***σ-algebra*** in X if these three conditions are satisfied.

1. $X \in \mathcal{M}$.
2. $X - A \in \mathcal{M}$ for every $A \in \mathcal{M}$.
3. $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ for every countable collection $\{A_n : n = 1, 2, 3, \ldots\}$ of elements in $\mathcal{M}$ [2]

A non-empty collection S of subsets of a set X is called a ***semi-ring*** provided whenever A and B belong to S, then $A \cap B$ also belongs to S and there is a finite disjoint collection $\{C_k\}_{k=1}^{n}$ of sets in S for which, $A - B = \bigcup_{k=1}^{n} C_k$ [4].

By a ***measurable space***, we mean a couple (X, $\mathcal{M}$) consisting of a set X and a σ-algebra $\mathcal{M}$ of subsets of X. A subset E of X is called ***measurable*** or ***measurable with respect to*** $\mathcal{M}$ provided E belongs to $\mathcal{M}$ [3].

A ***measure*** μ on a measurable space (X, $\mathcal{M}$), is an extended real-valued non-negative set function; $\mu : \mathcal{M} \to [0, \infty]$ for which $\mu(\emptyset) = 0$, and which is ***countably additive*** in the sense that for any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k). \qquad [1]$$

By a ***measure space*** $(X, \mathcal{M}, \mu)$, we mean a measurable space $(X, \mathcal{M})$ together with a measure μ defined on $\mathcal{M}$ [8].

Premeasure: A finitely additive, countably monotone set function s, defined on a ring $\mathcal{O}$ or algebra $\mathcal{A}$ is called a ***premeasure*** on $\mathcal{O}$ or $\mathcal{A}$ and $s(\emptyset) = 0$ [3].

Outer measure: A set function $\mu^* : 2^X \to [0, \infty]$ is called an ***outer measure*** provided $\mu^*(\emptyset) = 0$ and μ^* is countably monotone [8].

Theorem 1: Let μ^* be an outer measure on 2^X. Then the collection $\mathcal{M}$ of sets that are measurable with respect to μ^* is a σ-algebra. If μ is the restriction of μ^* to $\mathcal{M}$, then $(X, \mathcal{M}, \mu)$ is a complete measure space [9].

Carathéodory's definition: For an outer measure $\mu^* : 2^X \to [0, \infty]$, we call a subset E of X measurable (with respect to μ^* or μ^*-measurable) provided for every subset A of X, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ [1], [4].

An outer measure m^* on X is called a metric outer measure or an outer measure in the sense of Carathéodory (Carathéodory outer measure) if, $m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2)$ whenever $d(A_1, A_2) > 0$ [13].

Carathéodory's Extension Theorem: Let μ be a measure on an algebra $\mathcal{A}$ and μ^* the outer measure induced by μ. Then the restriction $\overline{\mu}$ of μ^* to the μ^*-measurable sets is an extension of μ to a σ-algebra containing $\mathcal{A}$. If μ is σ-finite, then $\overline{\mu}$ is the only measure on the smallest σ-algebra containing $\mathcal{A}$ which is an extension of μ [6], [9].

The Carathéodory-Hahn Theorem: Let $\mu : S \to [0, \infty]$ be a premeasure on a semiring S of subsets of X. Then the Carathéodory measure $\overline{\mu}$ induced by μ is an extension of μ. Furthermore, if μ is σ-finite, then so is $\overline{\mu}$ and $\overline{\mu}$ is the unique measure on the σ-algebra of μ^*-measurable sets that extends μ [8].

RESULTS

HAUSDORFF MEASURE

The Hausdorff measure is an example of measures constructed from a Carathéodory outer measure known as the Hausdorff outer measure.

Choose $\alpha > 0$, and let A be any subset of $\mathbb{R}^n$. Given $\varepsilon > 0$, let

$$H_\alpha^{(\varepsilon)}(A) = \inf \sum_k \delta(A_k)^\alpha, \qquad (2)$$

where $\delta(A_k)$ denotes the diameter of A_k and the infimum is taken over all countable collections $\{A_k\}$ such that $A \subseteq \bigcup A_k$ and $\delta(A_k) < \varepsilon$ for all k [13].

$$\text{Let } H_\alpha(A) = \lim_{\varepsilon \to 0} H_\alpha^{(\varepsilon)}(A). \qquad (2b)$$

Theorem: For $\alpha > 0$, H_α is a Carathéodory outer measure on $\mathbb{R}^n$ [13].

By the above theorem, H_α as defined by equation (2b) is a Carathéodory outer measure and thus by theorem 1, the collection $\mathcal{M}$ of sets that are measurable with respect to H_α is a σ-algebra. The restriction of H_α to $\mathcal{M}$ is referred to as the Hausdorff measure.

PRODUCT MEASURE

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, v)$ be two reference σ-finite measure spaces. We consider the Cartesian product $X \times Y$ of X and Y. If $A \subseteq X$ and $B \subseteq Y$, we call $A \times B$ a rectangle. If in addition $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then we call $A \times B$ a measurable rectangle [1].

Lemma 1: Let $\{A_k \times B_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable rectangles whose union is also a measurable rectangle $A \times B$. Then,

$$\mu(A) \cdot v(B) = \sum_{k=1}^{\infty} \mu(A_k) \times v(B_k). \qquad [8]$$

Proposition 1: Let $\mathcal{R}$ be the collection of measurable rectangles in $X \times Y$ and for a measurable rectangle $A \times B$, define $m(A \times B) = \mu(A) \cdot v(B)$. Then $\mathcal{R}$ is a semiring and $m : \mathcal{R} \to [0, \infty]$ is a premeasure [8].

Lemma 1 is primarily used in the proof of Proposition 1.

Proposition 1 allows us to invoke the Carathéodory-Hahn theorem in order to make the following definition of product measures.

Definition: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, v)$ be measure spaces. $\mathcal{R}$ the collection of measurable rectangles contained in $X \times Y$ and m the premeasure defined on $\mathcal{R}$ by $m(A \times B) = \mu(A) \cdot v(B)$ for $A \times B \in \mathcal{R}$ [8].

By the ***product measure*** $m = \mu \times v$, we mean the Carathéodory extension of the premeasure $m : \mathcal{R} \to [0, \infty]$ defined on the σ-algebra of $(\mu \times v)^*$-measurable subsets of $X \times Y$ as stipulated in the Carathéodory-Hahn theorem.

RADON MEASURE

Definition: Let (X, $\mathcal{T}$) be a topological space. We call a measure μ on the Borel σ-algebra $\mathcal{B}(X)$ a ***Borel measure*** provided every compact subset of X has finite measure. A Borel measure μ is called a ***Radon measure*** provided,

(i) for each Borel subset E of X, $\mu(E) = \inf\{\mu(U) : U \text{ a neighborhood of } E\}$ (***Outer regularity***),

(ii) for each open subset $\mathcal{O}$ of X, $\mu(\mathcal{O}) = \sup\{\mu(K)$: K a compact subset of $\mathcal{O}\}$(***Inner Regularity***). [8]

To arrive at the definition above, a natural place to begin is to consider premeasures $\mu : \mathcal{T} \to [0, \infty]$ defined on the topology $\mathcal{T}$ and consider the Carathéodory outer measure induced by μ.

Lemma 2: Let (X, $\mathcal{T}$) be a topological space, $\mu : \mathcal{T} \to [0, \infty]$ a premeasure, μ^* the outer measure induced by μ such that for any subset E of X, $\mu^*(E) = \inf\{\mu(U) : U$ an open neighborhood of $E\}$. (3)
Then, E is μ^*-measurable if, and only if,

$$\mu^*(\mathcal{O}) \geq \mu^*(\mathcal{O} \cap E) + \mu^*(\mathcal{O} - E) \quad (4)$$

for each open set $\mathcal{O}$ for which $\mu(\mathcal{O}) < \infty$ [8].

Proposition 2: Let (X, $\mathcal{T}$) be a topological space, $\mu : \mathcal{T} \to [0, \infty]$ a premeasure. Assume that for each open set $\mathcal{O}$ for which $\mu(\mathcal{O}) < \infty$,

$$\mu(\mathcal{O}) = \sup\{\mu(U) : U \text{ open and } \overline{U} \subset \mathcal{O}\} \quad (5)$$

Then every open set is μ^*-measurable and the measure
$\mu^* : \mathcal{B}(X) \to [0, \infty]$ is an extension of μ [8].

Lemma 2 defines a premeasure μ on $\mathcal{T}$, and proceeds to to define the outer measure μ^* on X via μ in a form that endows μ^* with the property of outer regularity.

Proposition 2 asserts that the restriction of μ^* as defined by Lemma 3 to the Borel σ-algebra is a measure and an extension of μ, and Proposition 2 also endows all open sets with inner regularity.
Thus, $\mu^* : \mathcal{B}(X) \to [0, \infty]$ is a Radon measure.

LEBESGUE MEASURE ON $\mathbb{R}^n$

We call a point in $\mathbb{R}^n$ an ***integral point*** provided each of its coordinates is an integer and for a bounded interval I in $\mathbb{R}^n$, we define its ***integral count***, $\mu^{integral}(I)$, to be the number of integral points in I [8].

Lemma 3: For each $\varepsilon > 0$, define the ε-dilation, $T_\varepsilon : \Re^n \to \Re^n$ by $T_\varepsilon(x) = \varepsilon \cdot x$. Then for each bounded interval I in $\mathbb{R}^n$,

$$\lim_{\varepsilon \to \infty} \frac{\mu^{integral}(T_\varepsilon(I))}{\varepsilon^n} = vol(I).$$

Proposition 3: The set function volume, $vol : \mathcal{I} \to [0, \infty)$, is a premeasure on the semi-ring $\mathcal{I}$ of bounded intervals in $\mathbb{R}^n$ [8].

Outline of Proof for Proposition 3:

- We first show that $\mathcal{I}$ is a semi-ring.
- Next, we show that vol is a premeasure on $\mathcal{I}$; monotonicity follows from the monotonicity of the integral count.
- Finite additivity of vol follows from the finite additivity of the integral count.

The outer measure μ_n^* induced by the premeasure, volume, on the semi-ring of bounded intervals in $\mathbb{R}^n$ is called the Lebesgue outer measure on $\mathbb{R}^n$.

The collection of μ_n^*-measurable sets is denoted by $\mathcal{L}^n$ and called the Lebesgue measurable sets. The restriction of μ_n^* to $\mathcal{L}^n$ is called the Lebesgue measure on $\mathbb{R}^n$ or n-dimensional Lebesgue measure and denoted by μ_n. The existence of μ_n is guaranteed by the Carathéodory-Hahn theorem.

LEBESGUE-STIELTJES MEASURE

To construct the Lebesgue-Stieltjes outer measure, we consider any fixed function f which is finite and monotonically increasing on $(-\infty, +\infty)$.

- For each half-open finite interval of the form $(a, b]$, let $m(a, b] = m_f((a, b]) = f(b) - f(a)$.

- If A is a non-empty subset of $\mathbb{R}$, let
$$m^*(A) = m_f^*(A) = \inf \sum_{k=1}^{\infty} m(a_k, b_k], \qquad (1)$$
where the infimum is taken over all countable collections $\{(a_k, b_k]\}$ such that $A \subseteq \bigcup (a_k, b_k]$. Further, define $m^*(\emptyset) = 0$ [13].

Theorem:

m^* is a Carathéodory outer measure on $\mathbb{R}$ [13].

Outline of Proof:

- We first show that $m^* \geq 0$ and $m^*(\emptyset) = 0$.

- Next, we show monotonicity, that is, if $A_1 \subseteq A_2$, then $m^*(A_1) \leq m^*(A_2)$.

- We show that m^* is sub-additive, that is, if $\{A_j\}_{j=1}^{\infty}$ is a collection of non-empty subsets of $\mathbb{R}$ and such that $A = \bigcup A_j$, then A $\subseteq \bigcup_{k,j} (a_k^j, b_k^j]$, and $m^*(A) \leq \sum m^*(A_j)$.

- Finally, we show that m^* is a Carathéodory outer measure, that is, if $d(A_1, A_2) > 0$, then, $m^*(A_1) + m^*(A_2) = m^*(A_1 \cup A_2)$

The theorem above asserts that, m^* as defined by equation (1) is an outer measure, and thus by Theorem 1, there is an associated σ-algebra, which is the collection of all subsets which are measurable with respect to m^*.

m_f^* is called the Lebesgue-Stieltjes outer measure corresponding to f, and its restriction to those sets which are m_f^*-measurable is called the Lebesgue-Stieltjes measure.

CONCLUSION

We used the Carathéodory-Hahn theorem, Carathéodory's Extension theorem and Theorem 1 in most of our constructions. Any such construction yields a complete measure space.

There are other ways of constructing measures. For example; a measure defined on the Borel σ-algebra, $\mathcal{B}$, is called a Borel measure, that is, if we restrict, say, the Lebesgue measure, m, on $\mathbb{R}$ to the Borel σ-algebra, $\mathcal{B}(\Re)$, we get a measure space $(\Re, \mathcal{B}(\Re), m)$. But this measure space lacks the property of completeness.

References

[1] Robert G. Bartle, *The Elements of Integration and Lebesgue Measure*, Wiley and Sons Inc. 1995.

[2] Donald L. Cohn, *Measure Theory*, Birkhàuser, 1980.

[3] Jewgeni H. Dshalalow, *Real Analysis, An Introduction to the Theory of Real Functions and Integration*, Chapman & Hall/CRC, 2001.

[4] D. H. Fremlin, *Measure Theory, volume 1*, Torres Fremlin, 2000.

[5] Orrin Frink Jr. *Jordan Measure and Riemann Integration*, The Annals of Mathematics, Second Series, Vol. 34, No. 3 (Jul., 1933), pp. 518-526.

[6] Paul R. Halmos, *Measure Theory*, Springer-Verlag New York Inc., 1974.

[7] Stanisław Łowjasiewicz, translated by G. H. Lawden, *An Introduction to the Theory of Real Functions*, Wiley and Sons Ltd., 1988.

[8] H. L. Royden and P. M. Fitzpatrick, *Real Analysis*, Fourth edition, Prentice Hall, 2010.

[9] H. L. Royden, *Real Analysis*, Third edition, Macmillan, 1988.

[10] Walter Rudin, *Principles of Mathematical Analysis*, Third edition, McGraw Hill, 1976.

[11] Walter Rudin, *Real and Complex Analysis*, International student edition, McGraw Hill, 1970.

[12] H. V. Weizäcker *Basic Measure Theory*, Revised translation.

[13] R. L. Wheeden and A. Zygmund, *Measure and Integral, An Introduction to Real Analysis*, Marcel Dekker Inc. 1977.

[14] W. P. Ziemer, *Modern Real Analysis*, Department of Mathematics, Indiana University, Indiana.

[15] Encyclopedia Britannica Online, s.v. "probability theory," March 14, 2013, http://www.britannica.com/EBchecked/topic/477530/probability-theory.