Plane Symmetric Vacuum Cosmological Model with a Special Form of Deceleration Parameter in $f(R)$ Theory of Gravity

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Abstract: In this paper we have obtained vacuum solutions of the plane symmetric space-time in $f(R)$ gravity. The general solutions of the field equations of plane symmetric space-time have been obtained under the assumption of special form of deceleration parameter**.** The physical and geometrical aspect of the model is also discussed.

Keywords: $f(R)$ gravity, plane symmetric space-time, special form of deceleration parameter.

1. INTRODUCTION:

Among the various modification of general relativity, the *f(R)* theory of gravity is treated most seriously during the last decade. The *f*(*R*) theory of gravity has also been helpful in describing the evolution of the universe. It provides a natural gravitational alternative to dark energy. Carroll *et al.*(2004) explained the presence of a late time cosmic acceleration of the universe in *f*(*R*) gravity. Bertolami *et al.*(2007) have proposed a generalization of *f*(*R*) modified theories of gravity by including in the theory an explicit coupling of an arbitrary function of the Ricci scalar *R* with the matter Lagrangian density L_m. As a result of the coupling, the motion of the massive particles is non-geodesic, and an extra force, orthogonal to the four-velocity, arises. The connections with Modified Newtonian Dynamics (MOND) and the Pioneer anomaly were also explored. This model was extended to the case of the arbitrary couplings in both geometry and matter by Harko (2008). The astrophysical and cosmological implications of the non-minimal coupling matter-geometry coupling were extensively investigated by Harko (2010). The Palatini formulation of the non-minimal geometry-coupling models was considered by Harko *et al*.(2010). Harko & Lobo (2010) proposed a maximal extension of the Hilbert-Einstein action, by assuming that the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and of the matter Lagrangian L_m. The $f\,(R)$ gravity provides a very natural unification of the early-time inflation and late-time acceleration. It describes the transition from deceleration to acceleration in the evolution of the universe (Nojiri and Odintsov 2007, 2008). Over the past few years, Many works are available in literature (Capozziello and Francaviglia 2008; Abdalla *et al*. 2005; Nojiri and Odintsov 2007b; Harko 2008) addressing the well-known issues of stability (Dolgov and Kawasaki 2003), singularity problem (Frolov 2008), solar system test (Chiba 2003b), etc. The general schemes for modified gravity reconstruction from any realistic FRW cosmology have been discussed by Nojiri and Odintsov (2006). It seems that $f\,(R)$ gravity models pass all known observational local test currently (Elizalde *et al*. 2010, 2011; Nojiri and Odintsov 2011). Shamir (2009, 2010a, 2010b), Sharif and Zubair (2010a), Shamir (2010), Sharif and Kausar (2011a, 2011b, 2011c), and Akta¸s *et al.* (2012) have studied anisotropic models in $f\,(R)$ theory. Cappozziello *et al.* (2009), Felice & Tsujikawa (2010), Zhai & Liu (2011) have studied various aspects of $f\,(R)$ theory of gravity in detail. Recently, Singh *et al* (2013) have studied Functional form of $f\,(R)$ with power-law expansion in anisotropic model.

These are the motivations to consider *f*(*R*) theory of gravity by large number of researchers. In this paper we have considered the plane symmetric space-time in $f\,(R)$ gravity. The general solutions of the field equations of plane symmetric space-time have been obtained under the assumption of special form of deceleration parameter**.** The physical and geometrical aspects of the model are also discussed.

2. *f*(*R*) THEORY OF GRAVITY:

We know that the $f(R)$ theory of gravity is the generalization of general relativity.

The action for $f(R)$ theory of gravity is represented by

$$S=\int\sqrt{-g}\left(\frac{1}{16\pi G}f(R)+L_m\right)d^4x\ . \tag{2.1}$$

Here $f(R)$ is a general function of the Ricci scalar R and L_m is the matter Lagrangian. One should note that above action is obtained just by replacing R by $f(R)$ in the standard Einstein-Hilbert action expression.

Now, by varying the action given by equation (2.1) with respect to the metric ($g_{\mu\nu}$), we get the corresponding field equations of *f(R)* gravity as

$$F(R)R_{\mu\nu}-\frac{1}{2}f(R)g_{\mu\nu}-\nabla_\mu\nabla_\nu F(R)+g_{\mu\nu}\Box F(R)=\kappa T_{\mu\nu}, \tag{2.2}$$

where $F(R)=df(R)/dR$, $\Box\equiv\nabla^\mu\nabla_\mu$, ∇_μ is the covariant derivative, $T_{\mu\nu}$ is the standard matter energy-momentum tensor derived from the Lagrangian L_m and $\kappa\,(=8\pi G/c^4)$ is the coupling constant in gravitational units.

Now contracting the field equations (2.2), we get

$$F(R)R-2f(R)+3\Box F(R)=\kappa T\,, \tag{2.3}$$

and in vacuum (i.e. for $T=0$), we have

$$F(R)R - 2f(R) + 3\Box F(R) = 0. \tag{2.4}$$

From equation (2.4), we get

$$f(R) = \frac{F(R)R}{2} + \frac{3}{2}\Box F(R). \tag{2.5}$$

The equation (2.5) gives an important relationship between $f(R)$ and $F(R)$ which will be used to simplify the field equations and to evaluate $f(R)$ also.

3. METRIC AND THE FIELD EQUATIONS:

In view of the importance of the plane symmetry, we consider the line element in plane symmetric form [Zhang & Noh 2009, Setare & Momeni 2010, Shen & Zhao 2012] as

$$ds^2 = dt^2 - A^2(dx^2 + dy^2) - B^2 dz^2, \tag{3.1}$$

where A and B are functions of the cosmic time t only.

The Ricci scalar for the line element (3.1) has value

$$R = -2\left[2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}^2}{A^2} + 2\frac{\dot{A}\dot{B}}{AB}\right], \tag{3.2}$$

where overhead dot (˙) represents derivative with respect to time t.

Using equation (2.5) in the vacuum field equations (2.2) (i.e. for $T = 0$), we have

$$\frac{1}{4}[F(R)R - \Box F(R)] = \frac{F(R)R_{\mu\nu} - \nabla_\mu \nabla_\nu F(R)}{g_{\mu\nu}}. \tag{3.3}$$

Since the metric (3.1) depends only on t, one can view (3.3) as a set of differential equations for $A(t)$, $B(t)$ and $F(t)$.

It follows from equation (3.3) that the combination

$$K_\mu \equiv \frac{F(R)R_{\mu\mu} - \nabla_\mu \nabla_\mu F(R)}{g_{\mu\mu}} \tag{3.4}$$

is independent of the index μ and hence $K_\mu - K_\nu = 0$ for all μ and ν.

Here K_μ is just a notation for the traced quantity.

The field equations in $f(R)$ gravity for the metric (3.1) with the help of equation (3.4)

[for $K_0 - K_1 = 0$, $K_0 - K_2 = 0$, and $K_0 - K_3 = 0$ respectively] are given by

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{\dot{A}^2}{A^2} - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{A}\dot{F}}{AF} + \frac{\ddot{F}}{F} = 0 \quad , \tag{3.5}$$

$$2\frac{\ddot{A}}{A} - 2\frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}\dot{F}}{BF} + \frac{\ddot{F}}{F} = 0 \quad , \tag{3.6}$$

where overhead dot ($\cdot$) denotes derivative with respect to time *t*.

4. SOLUTIONS OF THE FIELD EQUATIONS:

The field equations (3.5) and (3.6) are two non-linear differential equations with three unknowns A, B and F . In order to solve this system completely, we use a special form of deceleration parameter defined by Singha and Debnath (2009) as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -1 + \frac{\alpha}{1+a^\alpha}, \tag{4.1}$$

where $\alpha > 0$ is a constant and a is scale factor of the universe.

After solving equation (4.1) one can obtain the mean Hubble parameter H as

$$H = \frac{\dot{a}}{a} = k(1 + a^{-\alpha}), \tag{4.2}$$

where $k > 0$ is a constant of integration.

On integrating equation (4.2), we obtain the mean scale factor as

$$a = (e^{k\alpha t} - 1)^{1/\alpha}. \tag{4.3}$$

In view of space time (3.1), the spatial volume V and average scale factor a will be

$$V = A^2 B \quad \text{and} \quad a = (A^2 B)^{1/3} . \tag{4.4}$$

The mean Hubble parameter H will be

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(H_x + H_y + H_z\right) , \tag{4.5}$$

where $H_x = H_y = \frac{\dot{A}}{A}$, $H_z = \frac{\dot{B}}{B}$ are the directional Hubble parameters in the directions of x, y and z axes respectively.

Now, subtracting equation (3.6) from equation (3.5), we get

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) + \frac{\dot{F}}{F}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = 0 , \tag{4.6}$$

After solving equation (4.6), one can write the metric functions A and B explicitly as

$$A = c_2^{1/3}\, a \, \exp\left(\frac{c_1}{3}\int \frac{dt}{a^3 F}\right) , \tag{4.7}$$

$$B = c_2^{-2/3}\, a \, \exp\left(-\frac{2c_1}{3}\int \frac{dt}{a^3 F}\right) , \tag{4.8}$$

where c_1 and c_2 are constants of integration.

Now, we use the power-law to solve the integral part in the above equations. The power-law relation between scale factor and scalar field has already been used by Johri and Desikan (1994). Kotub Uddin *et al.*(2007) , Sharif & Shamir (2009) have established a result in the context of $f(R)$ gravity which shows that $F \propto a^m$.

Thus, using power-law relation between F and a, we have

$$F = l\, a^m , \tag{4.9}$$

where l is the constant of proportionality and m is any integer.

Using equations (4.3) and (4.9) for $k = 1, \alpha = 2$ and $m = -2$ in the equations (4.7) and (.4.8), we obtain the scale factors as

$$A = c_2^{1/3}(e^{2t}-1)^{\frac{1}{2}} \exp\left[\frac{c_1}{3l}\tan^{-1}\left(e^{2t}-1\right)^{\frac{1}{2}}\right], \tag{4.10}$$

$$B = c_2^{-2/3}(e^{2t}-1)^{\frac{1}{2}} \exp\left[-\frac{2c_1}{3l}\tan^{-1}\left(e^{2t}-1\right)^{\frac{1}{2}}\right]. \tag{4.11}$$

where c_1 and c_2 are constants of integration and l is the constant of proportionality.

***Some Physical Properties*:**

Using equations (4.10) and (4.11), the directional Hubble parameters in the directions of x, y and z-axis are found to be

$$H_x = H_y = \frac{e^{2t}}{(e^{2t}-1)} + \frac{c_1}{3l\left(e^{2t}-1\right)^{\frac{1}{2}}}, \tag{4.12}$$

$$H_z = \frac{e^{2t}}{(e^{2t}-1)} - \frac{2c_1}{3l\left(e^{2t}-1\right)^{\frac{1}{2}}}. \tag{4.13}$$

The mean Hubble parameter H is found to be

$$H = \frac{e^{2t}}{(e^{2t}-1)}. \tag{4.14}$$

Using equations (4.10) and (4.11) in equation (4.4), the volume V of the universe is given by

$$V = \left(e^{2t}-1\right)^{\frac{3}{2}}. \tag{4.15}$$

The expansion scalar $\theta = 3H$ is given by

$$\theta = \frac{3e^{2t}}{(e^{2t}-1)}. \tag{4.16}$$

The mean anisotropy parameter Δ of the expansion is define as

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2,$$

where $H_i\left(i = 1, 2, 3\right)$ represent the directional Hubble parameters.

The anisotropy parameter Δ of the expansion is found to be

$$\Delta = 1 + \frac{2c_1^2}{9l^2}\frac{(e^{2t} - 1)}{e^{4t}}. \tag{4.17}$$

The shear scalar is define as $\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{3} H_i^2 - 3H^2\right)$ and found to be

$$\sigma^2 = \frac{c_1^2}{3l^2(e^{2t} - 1)} + \frac{3e^{4t}}{2(e^{2t} - 1)^2}. \tag{4.18}$$

The deceleration parameter is define as $q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1$ and found to be

$$q = \frac{2}{e^{2t}} - 1. \tag{4.19}$$

Using equations (4.10) and (4.11) in equation (3.2), the Ricci scalar for Bianchi type-I model is given by

$$R = 2\left[\frac{c_1(2e^{2t} + 1)}{3l(e^{2t} - 1)^{3/2}} + \frac{(2c_1^2 + 45l^2 e^{2t})}{9l^2\,(1 - e^{2t})}\right]. \tag{4.20}$$

Using equation (2.5), we obtain the function of Ricci scalar i.e. $f(R)$ as

$$f(R)=\frac{1}{2\left(e^{2t}-1\right)^{3}}\left[6l\left(2-e^{2t}\right)+l\left(e^{2t}-1\right)^{2}R\right]. \qquad (4.21)$$

5. DISCUSSION:

(i) From figure 1, one can observe that the spatial volume V vanishes at $t=0$. It expands exponentially as time t increase and becomes infinitely large as $t\rightarrow\infty$.

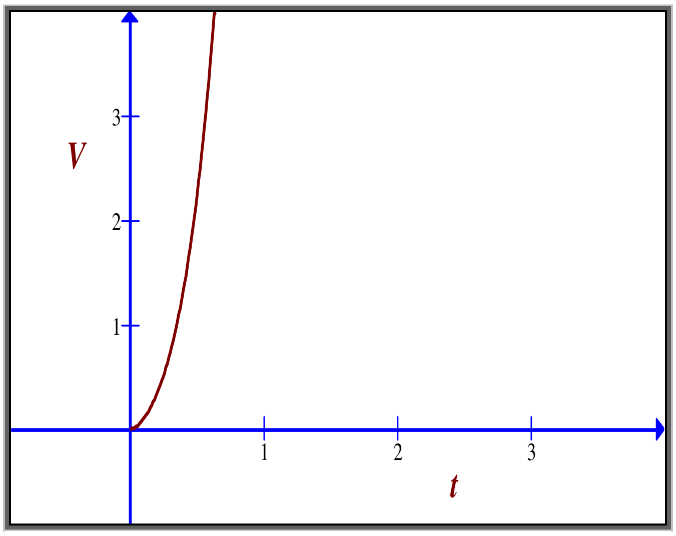

Figure (1): The variation of V vs. t for $k=1, \alpha=2$.

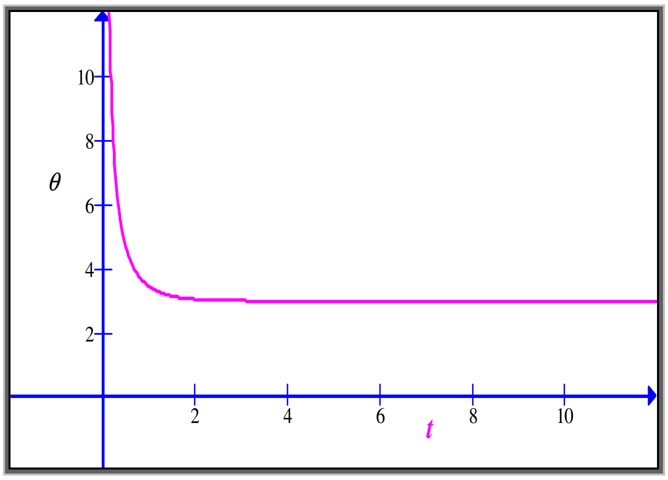

Figure (2): The variation of θ vs. t for $k = 1, \alpha = 2$.

From figure (2), it is observed that the expansion scalar θ starts with infinite value at $t = 0$ and then rapidly becomes constant after some finite time.

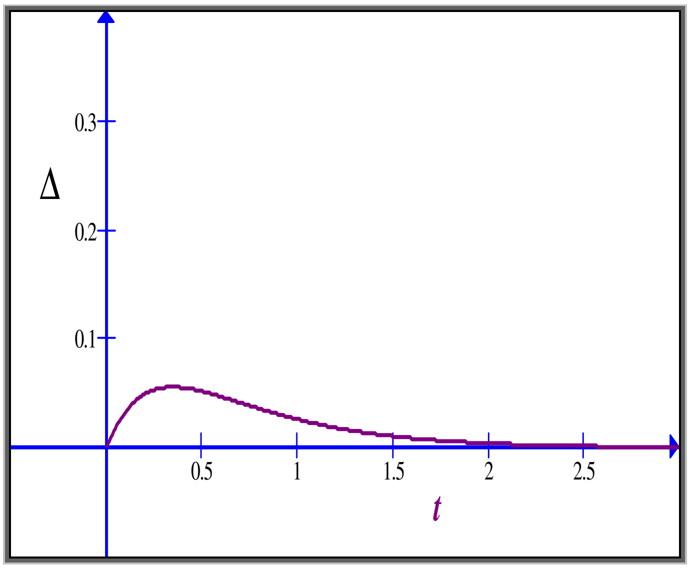

Figure (3): The variation of Δ vs. t for $k = 1, \alpha = 2$.

From figure (3), it is observed that anisotropy Δ increases as time increases and then quickly decreases to zero after some time and remains zero after some finite time.

Hence, the model reaches to isotropy after some finite time which matches with the recent observation as the universe is isotropic at large scale.

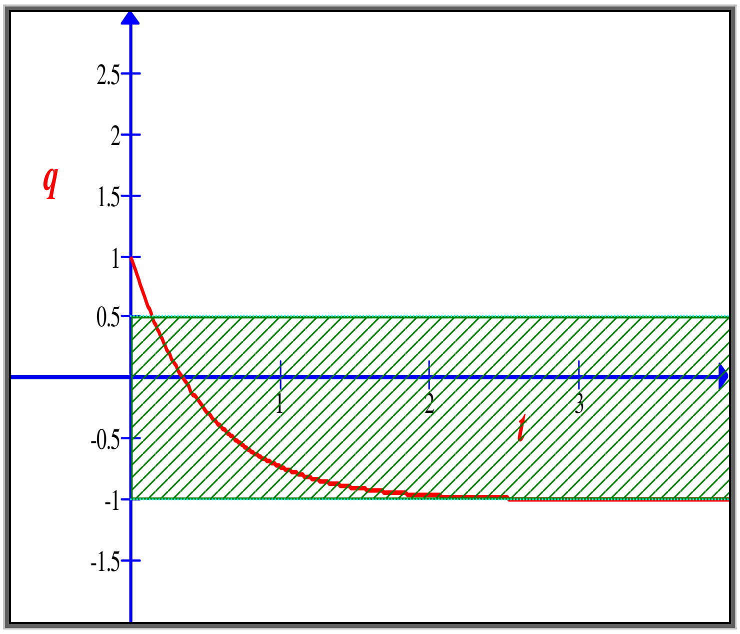

Figure (4): The variation of q vs. t for $\alpha = 2$.

The deceleration parameter q varies from $+1$ to -1 as shown in above figure (4). The deceleration parameter q is in the range $-1 \le q \le 0.5$ (shaded region in the fig.4).

6. CONCLUSION:

(i) It should be noted that the solutions given by equations (4.10) & (4.11) are different to the solutions obtained by Sharif & Shamir (2009) ,whereas, they are similar to the solutions obtained by Reddy *et al.* (2014).

(ii) It is interesting to observe that, in this case, we get, the deceleration parameter q in the range $-1 \le q \le 0.5$ which matches with the observations made by Riess *et al.* (1998) and Perlmutter *et al.* (1999) and the present day universe is undergoing accelerated expansion. It also shows that the universe accelerates after an epoch of deceleration.

(iii) This idea can be explored much in the forthcoming papers.

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