Two-Body Dirac Theory

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Zitterbewegung is the quiver motion with frequency $\omega = \frac{2mc^2}{\hbar}$ and Compton-wavelength amplitude $\lambda = \frac{\hbar}{mc}$ discovered by Schroedinger in his 1930 solution of the time-dependent Dirac equation for a free electron. The origin of Zitterbewegung is the interference between positive-energy and negative-energy states. Zitterbewegung is predicted to exist for both free and bound electrons provided one discards Dirac's hypothesis, known as hole theory, whereby positive-energy electrons are forbidden from occupying negative-energy states such that Zitterbewegung is suppressed. The positive-energy spectra are identical whether one uses Dirac's hypothesis, in which Zitterbewegung is suppressed, or whether one uses the solution in which electrons are not restricted from simultaneously occupying both positive-energy and negative-energy states such that Zitterbewegung is not suppressed. Thus both the restricted and unrestricted solutions are confirmed by spectroscopic experimental observation such that new experiments are motivated to discriminate between the restricted and unrestricted solutions. Restricted Dirac theory is also validated by electron-positron pair creation and annihilation experiments. In this paper we look for pair states in the negative-energy region of the Dirac spectrum in order to understand if the positive-energy negative-energy interference solution, which correctly predicts the positive-energy spectrum, is also correct in the description of pair creation and annihilation.

I. Introduction and general considerations

A considerable literature exists on different electrodynamical theories proposed for the calculation of the radiative properties of matter, for which, in order better to orient the reader to the subject matter of this paper, we give a brief review as follows. The quantization of the classical electromagnetic field, which founded quantum electrodynamics (QED), was carried out by Dirac in 1927 [1]. A review of the quantized radiation field (QRF), as it is called, and its use in the calculation of radiative spontaneous emission and the Lamb shift is given by Louisell [2]. The QRF may be criticized in the sense that its distribution of frequencies is unrelated to the electron's own distribution of frequencies and is therefore unbounded such that its use in the radiation-matter interaction Hamiltonian for the electron leads to an energy shift – Lamb shift – which diverges linearly in the photon frequency, ω. As explained in [2] and elsewhere the linear divergence is interpreted as a permanent radiant property of a free electron such that, when it is included or "added back" to the calculation for a bound electron which is "bare" or undressed by the radiation field in the original calculation, the linear divergence is exactly canceled. This procedure is known as mass renormalization because it contributes a radiative component to the electron's material mass. Although a logarithmic divergence in the photon frequency remains, use of a suitable cut off leads to results which agree quantitatively with experiment [2-3]. Notice that the linearly-divergent contribution to the mass of a free electron appears to be irremovable.

In order to gain a more satisfactory physical picture of the radiant aspect of the

electron, pioneers have presented formulations – the neoclassical theory of Edwin Jaynes and coworkers [4] and self-field quantum electrodynamics of Asim Barut and coworkers [5] – in which the electromagnetic vector potential is calculated from the electron's current. These theories were problematic either in a quantitative sense in Jaynes' case or in the sense of possible flaws in the use of Schroedinger theory to calculate the electron's current in Barut's case [6-7].

The quantization condition for the photon and electron [1-2] requires that an initial higher-energy state of the electron has zero photons and that a final lower-energy state of the electron has one photon. The radiative emission rate converges because it vanishes by destructive interference of the out-of-phase electron wave functions of the initial and final states unless $\hbar\omega = \Delta E_{fi}$, where ΔE_{fi} is the energy gap between the two states and $\hbar\omega$ is the photon energy. On the other hand emission of a photon from the ground state means that the photon must be re-absorbed by the same state leading to a closed photon loop in which the electron energy shift diverges as ω. Dirac's relativistic-electron theory [8] leads to further complications in interpretation since a set of negative-energy states lies below the nominal ground state such that radiative spontaneous emission from the ground state to a negative-energy state lying below it would occur, which is unobserved in nature. Dirac's hole interpretation that the negative-energy states are filled with electrons in which an absent electron or hole represents a positron avoids the unphysical prediction since a positive-energy electron is forbidden by Pauli's exclusion principle from falling into a negative-energy state. But Dirac's hole

theory also rules out the existence of Zitterbewegung, which arises from the interference between positive-energy and negative-energy states in observables in which an electron simultaneously occupies a superposition of positive-energy and negative-energy states. The recent observation of Zitterbewegung in a simulated electron experiment using a trapped-ion [9] suggests that hole theory, for all its success in positron physics, should be reexamined from the point of view of its reconciliation with Zitterbewegung. What is the ground state? A body of theory exists known as 4-space Dirac theory [10-11], whose principal motivation is the clear avoidance of a preferred frame of reference. In 4-space Dirac theory the positive-energy spectrum of states is identical to that of standard Dirac theory but the wave function comprises contributions from both electrons and positrons, which one may surmise is just a bound-state form of Zitterbewegung, although not identified as such likely owing to the fact that the original prediction of Zitterbewegung [12] was made for a free electron. Following Barut and coworkers [5,13] and others, it is necessary here to pursue a first-quantization approach in order to understand phenomena usually treated within second quantization.

Recent work [10] suggests that the negative-energy states do not lie empty below the ground state but rather actively participate with it to form a two-component ground-state configuration. If the negative-energy states do not lie empty below the positive-energy states, then the quantization rules for radiative spontaneous emission do not physically apply. In [10-11] the positive-energy spectrum is identical to that of standard Dirac theory, but the wave function exhibits

Zitterbewegung (or comprises contributions from both electrons and positrons in the post-hole language of [11]). But the original motivation and experimental confirmation of Dirac theory was the spectroscopic observation of atomic fine structure. *Thus standard Dirac theory and 4-space Dirac theory [10-11] are therefore both confirmed by spectroscopic experiments, such that the confirmation of wave-function Zitterbewegung predicted by 4-space theory requires experiments designed to probe the wave function and not the energy spectrum.* In short Dirac hole theory is incompatible with the experimental observation of Zitterbewegung, which exists if indeed the negative-energy states do not lie empty below the positive-energy states such that radiative spontaneous emission from the *nominally* positive-energy ground state cannot exist and therefore does not need to be blocked by the artifice of filling up the negative-energy levels with electrons, an absent electron from which represents, in an abstract sense, a positron.

It seems clear from the above discussion that, while the QRF is physically correct for radiative spontaneous emission, it has unphysical consequences for the radiative shift of energy levels, which is corrected in practical applications by using the physical argument of mass renormalization. Indeed in his original paper [1] Dirac *limits* the use of the QRF to the emission and absorption of radiation and the derivation of the Einstein A and B coefficients. But one can use the renormalization concept that an electron permanently has radiant properties which are therefore always present, such that the concept of a bare electron loses meaning. In standard QED this concept takes the form of continuously emitted

and reabsorbed photons by the quantum state of a free electron, whose mathematical implementation, as stated above, leads to the divergent shift linear in ω for a free electron – the divergence which, when included in the bound-electron calculation, cancels the divergent shift linear in ω for the bound electron. One may postulate that this concept can be realized by finding a first-quantized Lorentz-invariant relativisitic equation of motion which accounts for the radiant properties of the electron in the same way that Dirac's equation accounts for the material properties of the electron. A small literature using the concept of a photon equation of motion (EOM) already exists [14], but its applications appear to be confined to experiments in which the radiation-matter interaction is unimportant.

The concept of radiation as a permanent part of the quantum states of the electron is actually introduced in renormalization theory, as discussed above. But again the field-theoretic logic of the continuous emission and absorption of virtual photons by the *same quantum state in a closed photon loop* leads to a radiative correction to the electron's mass which diverges as ω [2]. The missing concept whose mathematical implementation would avoid this failure uses the logic that the quantum states of matter *exist simultaneously and permanently with the quantum states of radiation* such that the artificial boundary-value setup of virtual-photon emission and absorption is averted. Since the quantum states of matter are given by Dirac's equation, we require a supplemental wave equation to give the quantum states of radiation associated with the electron. While Dirac's equation accounts for the material properties of the electron, the supplemental radiation wave equation

may be considered to account for the radiant properties of the electron, as observed experimentally in the Lamb shift and the electron's anomalous magnetic moment.

In summary radiation-free matter does not exist in nature. But theoretical physics has evolved, reflecting the separate developments historically of mechanics and electrodynamics, into a radiation-free quantum theory of matter, a matter-free quantum theory of radiation, and a theory of the mutual interaction of radiation and matter. This piecemeal approach leads to an infinite energy for the Lamb shift and other "radiative corrections" of the electron in absence of the use of physical argument and mathematical adjustments to "renormalize" the theory in order to obtain a finite result which remarkably agrees with high accuracy with experimental observation. One may question however if perfect theoretical agreement with a specific set of experiments should be accepted with uncritical acclaim in absence of a theory which explains the source of the infinities and provides a divergence-free result. It is hard to imagine that renormalization theory with its mathematical recipes for the removal of divergent contributions could be a general theory of nature, not withstanding its high degree of accuracy.

Indeed one may say that particle fields for matter-free photons or photon-free electrons represent incomplete physical descriptions of these particles. This is the lesson which we may take away from Lamb's experiments, which demonstrate the existence of permanent radiative shifts in atomic energy levels, namely that radiation-free matter does not exist in nature such that a photon-free material particle field or a particle-free photon field, however neat and pleasing this it is

mathematically, is not a complete picture either of the material particle or of the photon. The renormalization scheme itself confirms this view since infinities are removed from radiation-matter calculations by postulating that photons are always present in the structure of a free electron such that when the free-electron radiative shift is added back to bound-electron calculations the unphysical infinities are removed.

In Section II of this paper an equation of motion for the radiant aspect of the electron is given. Then in Section II we investigate the possibility that a bound state might exist in the negative-energy region of the spectrum of the two-body Dirac equation such that annihilation may be interpreted as an ordinary bound-bound transition of matter rather than as the conversion of matter into light. Finally in Section III our conclusions are presented.

II. Equation of motion for the electron's radiant aspect

It is easy to propose a radiant equation of motion (REOM) for the electron once it is recognized that the electron's material equation of motion (MEOM), which is Dirac's equation, can be inferred from the scalar product of the electron's 4-momentum and a material 4-potential posited for the electron. This understanding of Dirac's equation suggests that a REOM can be inferred from the photon's 4-momentum and an electromagnetic 4-potential posited for the electron.

Recalling that the scalar product of 4-vectors is always Lorentz invariant [15], Dirac's equation can be derived by further elucidating the close relationship between Dirac's equation and the spinorial form of Maxwell's equation, which has been studied continuously since 1928 [16-19]. Dirac's equation can be inferred from the scalar product of the electron's 4-momentum and a material 4-potential, $(\Psi,\vec{X})$, posited for the electron as follows,

$$(i\hbar\frac{\partial}{c\partial t}-\frac{V}{c},i\hbar\vec{\nabla}+\frac{e}{c}\vec{A})\cdot(\Psi,\vec{X})=(i\hbar\frac{\partial}{c\partial t}-\frac{V}{c})\Psi+(i\hbar\vec{\nabla}+\frac{e}{c}\vec{A})\cdot\vec{X}=0., \qquad \text{(II-1)}$$

Using a carrier-wave expansions for $(\Psi,\vec{X})$ in order to isolate a dominant frequency component of the 4-potential we obtain,

$$\Psi=\Psi_{-}(\vec{r},t)e^{-i\omega t}+\Psi_{+}(\vec{r},t)e^{i\omega t} \qquad \text{(II-2a)}$$

$$\vec{X}=\vec{X}_{-}(\vec{r},t)e^{-i\omega t}+\vec{X}_{+}(\vec{r},t)e^{i\omega t}. \qquad \text{(II-2b)}$$

On substituting Eqs. (II-2) into Eq. (II-1) and separately setting the coefficients of the exponential factors equal to zero, I obtain,

$$(i\hbar\frac{\partial}{\partial t} - V - \hbar\omega)\Psi_+ + (i\hbar c\vec{\nabla} + e\vec{A})\cdot \vec{X}_+ = 0 \qquad \text{(II-3a)}$$

$$(i\hbar\frac{\partial}{\partial t} - V + \hbar\omega)\Psi_- + (i\hbar c\vec{\nabla} + e\vec{A})\cdot \vec{X}_- = 0. \qquad \text{(II-3b)}$$

Dirac's equation ,

$$(i\hbar\frac{\partial}{\partial t} - V - mc^2)\psi + c\vec{\sigma}\cdot(i\hbar\vec{\nabla} + e\vec{A})\xi = 0 \qquad \text{(II-4a)}$$

$$(i\hbar\frac{\partial}{\partial t} - V + mc^2)\xi + c\vec{\sigma}\cdot(i\hbar\vec{\nabla} + e\vec{A})\psi = 0, \qquad \text{(II-4b)}$$

follows immediately on setting $\hbar\omega = mc^2$, $\vec{X}_+ = \vec{\sigma}\Psi_-$, $\vec{X}_- = \vec{\sigma}\Psi_+$, $\Psi_+ = \psi$, and $\Psi_- = \xi$. The reader may verify that Eqs. (II-4) are indeed Dirac's equation in coupled first-order form where ψ and ξ are known in the literature as the large and small components of the Dirac solution. Notice that the electron's spin can be interpreted as the polarization of the vector component of its posited material 4-potential. Notice that no further proof of the Lorentz invariance of the wave equation itself is required since Eqs. (II-4) have been inferred directly from a scalar product of 4-vectors, which is always a Lorentz invariant [15]. As an example Dirac's equation for a hydrogen-like ion [20] is manifestly Lorentz invariant, but a fully relativistic Lorentz-invariant theory for two fermions is given by the Bethe-Salpeter equation [21]. As a further complication Coulomb's law for the interelectronic interaction is incompatible with Lorentz invariance such that it is represented field-theoretically by the exchange of virtual photons. If indeed future experiments show that Zitterbewegung is a real physical effect arising from the simultaneous occupancy

of both positive-energy and negative-energy states by an electron, then the Bethe-Salpeter equation describing a positron and electron should be appealed to for a proper description of annihilation and pair creation. This point is examined further in Section III.

While Eqs. (II-4) account for atomic fine structure and the anomalous Zeeman effect, whose spectroscopic observation was the motivation for Dirac's equation and its experimental confirmation, radiant properties of the electron also exist which are observed as a quantum electrodynamical shift of atomic energy levels and the electron's anomalous magnetic moment. It is assumed that an electro-magnetic 4-potential exists for the electron such that a REOM can be inferred from the Lorentz invariant found from the scalar product of the photon's 4-momentum and the electron's posited electromagnetic 4-potential thusly,

$$(\frac{\hbar}{c}\frac{\partial}{\partial t},\hbar\vec{\nabla}-\frac{e\hbar}{mc^2}\vec{E},\vec{H})\cdot(\Phi_v,\vec{A}_v)=\frac{\hbar}{c}\frac{\partial}{\partial t}\Phi_e+(\hbar\vec{\nabla}-\frac{e\hbar}{mc^2}\vec{E},\vec{H})\cdot\vec{A}_v=0, \qquad \text{(II-5)}$$

where the notation $\vec{E},\vec{H}$ means *either* the electric *or* the magnetic fields, *either* $\vec{E}$ *or* $\vec{H}$ respectively. The photon four-momentum is found from $\hbar$ times a 4-gradient, $(\frac{\partial}{c\partial t},\vec{\nabla}-\frac{e}{mc^2}\vec{E},\vec{H})$, whose scalar product with the electromagnetic 4-current, $(c(u+\int_0^t dt'\vec{j}\cdot\vec{E}),\vec{S})$, where $u=\frac{1}{8\pi}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})$ is the electromagnetic energy density and $\vec{S}=\frac{c}{4\pi}\vec{E}\times\vec{H}$ is the electromagnetic 3-current, gives the Lorentz-invariant electromagnetic continuity equation,

$$\frac{\partial u}{\partial t}+\vec{\nabla}\cdot\vec{S}+\vec{j}\cdot\vec{E}=0\,. \qquad \text{(II-6)}$$

This is simply the electromagnetic analog of writing the Lorentz-invariant material

continuity equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0, \tag{II-7}$$

as the scalar product of the known 4-gradient, $(\frac{\partial}{c\partial t}, \vec{\nabla})$, and the known material 4-current, $(c\rho, \vec{j})$. Notice that in the radiant-electron theory developed above the known 4-gradient is simply renormalized by the replacement $\vec{\nabla} \rightarrow \vec{\nabla} - \frac{e}{mc^2}\vec{E},\vec{H}$, which gives a Lorentz-invariant electromagnetic continuity equation since the scalar product of $\vec{E}$ or $\vec{H}$ with the electromagnetic 3-current, $\vec{S}$, vanishes. It is remarkable that a photon 4-momentum seems not to have been previously proposed in the literature.

As with the electron the photon scalar and vector potentials can be written in the form of carrier-wave expansions,

$$\Phi_\nu = \Phi_{\nu-} e^{-i\omega_\nu t} + \Phi_{\nu+} e^{i\omega_\nu t} \tag{II-8a}$$

$$\vec{A}_\nu = \vec{A}_{\nu-} e^{-i\omega_\nu t} + \vec{A}_{\nu+} e^{i\omega_\nu t}, \tag{II-8b}$$

from which on substituting Eqs. (II-8) into Eq. (II-5) and separately setting the coefficients of the exponential factors equal to zero, we obtain,

$$(\frac{1}{c}\frac{\partial}{\partial t} + i\frac{\omega_\nu}{c})\Phi_{\nu+} + (\vec{\nabla} - \frac{e}{mc^2}\vec{E},\vec{H}) \cdot \vec{A}_{\nu+} = 0 \tag{II-9a}$$

$$(\frac{1}{c}\frac{\partial}{\partial t} - i\frac{\omega_\nu}{c})\Phi_{\nu-} + (\vec{\nabla} - \frac{e}{mc^2}\vec{E},\vec{H}) \cdot \vec{A}_{\nu-} = 0. \tag{II-9b}$$

On setting $\Phi_{\nu+} = \xi_{E,H}$, $\vec{A}_{\nu+} = \vec{\sigma}\zeta_{E,H}$, $\Phi_{\nu-} = \zeta_{E,H}$, $\vec{A}_{\nu-} = \vec{\sigma}\xi_{E,H}$ we obtain the

Dirac form for the REOM,

$$\frac{\partial \xi_{E,H}}{c\partial t} + i\frac{\omega_v}{c}\xi_{E,H} + \vec{\sigma}\cdot(\vec{\nabla} - \frac{e}{mc^2}\vec{E},\vec{H})\xi_{E,H} = 0 \qquad \text{(II-10a)}$$

$$\frac{\partial \zeta_{E,H}}{c\partial t} - i\frac{\omega_v}{c}\zeta_{E,H} + \vec{\sigma}\cdot(\vec{\nabla} - \frac{e}{mc^2}\vec{E},\vec{H})\xi_{E,H} = 0. \qquad \text{(II-10b)}$$

Writing $\xi_{E,H} = e^{-i\omega t}\psi_{E,H}$ and $\zeta_{E,H} = e^{-i\omega t}\chi_{E,H}$ in Eqs. (II-10) we derive stationary equations for $\psi_{E,H}$ and $\chi_{E,H}$; then we eliminate the equation for $\chi_{E,H}$ in favor of a second-order equation for $\psi_{E,H}$,obtaining equations for the electric and magnetic photon wave functions which have the Helmholtz form,

$$\{\nabla^2 + \frac{\omega^2 - \omega_v^2}{c^2} - \frac{e}{mc^2}[\vec{\nabla}\cdot\vec{E} + 2\vec{E}\cdot\vec{\nabla} + i\sigma\cdot(\vec{\nabla}\times\vec{E}) - \frac{e}{mc^2}E^2]\}\psi_E = 0 \qquad \text{(II-11a)}$$

$$\{\nabla^2 + \frac{\omega^2 - \omega_v^2}{c^2} - \frac{e}{mc^2}[\vec{\nabla}\cdot\vec{H} + 2\vec{H}\cdot\vec{\nabla} + i\sigma\cdot(\vec{\nabla}\times\vec{H}) - \frac{e}{mc^2}H^2]\}\psi_H = 0, \qquad \text{(II-11b)}$$

where we have used the identity, $(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$.

Eq. (II-11b), for $\hbar\omega_v = 0$, was used in physical applications to calculate a *divergence-free* Lamb shift [22] and electron's anomalous magnetic moment [23].

Notice that in Eqs. (II-11) the electromagnetic fields and not the electromagnetic potentials occur such there is no question of a gauge dependence of matter-light interactions in the electron's REOM. The success of the use Eqs. (II-11) to calculate divergence-free radiative properties of matter [22-23] suggests that the concept of radiation as a permanent part of the structure of matter is a valid one. Recall that this is identically the concept of mass renormalization used in standard

QED used to remove infinite contributions to the electron's energy arising from unphysical logic that *first-quantized* states of matter exist which are totally free of radiation. As we have shown here it is possible to present a theory in which the electron does not exist in a bare or radiation-free state and whose material and radiant properties are described by a pair of relativistic, Lorentz-invariant first-quantized material and radiant EOM's respectively.

III. Pair creation and annihilation in two-body Dirac theory.

Toward a resolution of the paradox of the incompatibility of hole theory and Zitterbewegung [9,12] a two-body Dirac theory is used to calculate the spectrum of the positronium atom (Ps). Countless annihilation and pair-creation experiments are successfully interpreted using hole theory, which is part of Dirac's interpretation of the negative-energy states, namely that an absent electron in the set of negative-energy states filled with electrons represents a positron such that annihilation and pair-creation are interpreted as two-photon transitions downward to and upward from a hole state respectively. Hence a reconciliation of Zitterbewegung and hole theory requires that the two-fermion spectrum for an electron and positron contain both the known ground state and a negative-energy state with binding energy equal to $2mc^2$. To our knowledge two-body Dirac theory has heretofore not been used to search for a bound state lying below the nominal ground state.

Two-body Dirac theory previously has taken several forms. A two-body Dirac equation can be written down and solved [24], but it is not Lorentz invariant owing to the 6-space nature of the positron-electron Coulomb interaction. Alternatively the Bethe-Salpeter equation [21] is a fully relativistic, Lorentz-invariant equation for two fermions, but Lorentz invariance requires that the Coulomb interaction must be represented by the exchange of virtual photons. The force-carrier formalism for the interaction suggests that the Bethe-Salpeter equation is of limited usefulness for bound-state problems which require a nonperturbatively solution. Also it is possible to use Dirac's relativistic constraint mechanics to find an exact solution

for the singlet state of positronium [25]. More recently a dynamical theory is available [26] which preserves Lorentz invariance for a many-electron ensemble by using a Dirac equation for each electron in which the Coulomb interaction (using a two-electron example) is calculated in 4-space as $v_{12} = \frac{e^2}{|\vec{r} - \vec{s}(t)|}$, where $\vec{r}$ is the 3-space position of the electron whose Dirac equation one is solving and $\vec{s}(t)$ is the spin-dependent quantum trajectory of the second electron which is inferred from the Dirac current of the second electron using its Dirac equation, and conversely for second electron. Coulomb's law is given to us by classical electrodynamics as the inverse distance between two point electrons, and it is calculated in relativistic 4-space for use in individual Lorentz-invariant equations of motion without resort to the artifice of representing it by the exchange of virtual photons. Thus far this methodology has been tested only in the nonrelativistic limit to solve the electron exchange-correlation problem. It provides a natural physical explanation for the Pauli Exclusion Principle and Fermi-Dirac statistics by correlating the electron-electron Coulomb interaction with the spin state of each electron. The historical evolution of quantum mechanics to construct a single wave function for N electrons instead of constructing N wave equations of established correctness for N electrons is possibly a scholastic exercise. In fact it is inconsistent with classical mechanics, in which the special theory of relativity was formulated by Einstein, as a theory using N equations of motion for N bodies. We believe that the one-body 4-space approach has a compatibility with the special theory of relativity which is obscured by the mathematical complexity of the standard approach seeking a single wave

function for N bodies.

In this paper we want to use a relativistic quantum theory for two fermions which is Lorentz invariant but yet free of the mathematical complexity of the theories just adumbrated. Therefore the Hartree or self-consistent field model will be used here. Although the Coulomb potential between the two fermions is replaced by its quantum mean with respect to either fermion and is thus an approximate interaction potential, this procedure nevertheless preserves Lorentz invariance in the form of a one-body Dirac equation for each fermion.

It suffices to present the Dirac equation for one of the fermions, which is given for the electron by Eqs. (4). In the present application $\vec{A} = 0$. The Hartree potential is

$$V = -e^2 \int d\vec{r}' \frac{|\varphi(\vec{r}')|^2 + |\chi(\vec{r}')|^2}{|\vec{r} - \vec{r}'|}, \qquad \text{(III-1)}$$

where the quantum mean of the attractive electron-positron Coulomb potential, $v_{12} = -\frac{e^2}{|\vec{r} - \vec{r}'|}$, is given by Eq. (III-1) is calculated using the quantum density of the positron as given by the sum of large-component and small component densities shown in the numerator on the right side of Eq. (III-1). Similarly the Dirac equation for the positron interacting with the electron via the quantum mean of the electron-positron Coulomb potential using the quantum density of the electron can be written down and solved. Each Dirac equation is of course fully relativistic and Lorentz invariant, but it must be solved iteratively since the potentials depend on knowing the quantum densities for each fermion. It is found that 10 iterations are sufficient to achieve convergence in the energy as given by the operator identity

$$i\hbar\frac{\partial}{\partial t} = E\,.$$

Eqs. (II-4) are solved variationally, first by separating variables using $\psi(\vec{r}) = G_\kappa(r)\chi_{\kappa\mu}(\theta,\phi)$ and $\xi(\vec{r}) = iF_\kappa(r)\chi_{-\kappa\mu}(\theta,\phi)$ such that

$$E = mc^2\int_0^\infty drr^2G_\kappa^2(r) - mc^2\int_0^\infty drr^2F_\kappa^2(r) + \int_0^\infty drr^2V(r)[G_\kappa^2(r)+F_\kappa^2(r)] + \hbar c\{\int_0^\infty drr^2[F_\kappa(r)G_\kappa' - G_\kappa(r)F_\kappa' + 2\frac{\kappa}{r}F_\kappa(r)G_\kappa]\} \qquad \text{(III-2)}$$

where the identities, $\vec{\sigma}\cdot\vec{\nabla} = \vec{\sigma}\cdot\hat{r}(\frac{\partial}{\partial r} - \frac{\vec{\sigma}\cdot\vec{\ell}}{r})$, $\vec{\sigma}\cdot\hat{r}\chi_{\kappa\mu}(\theta,\phi) = -\chi_{-\kappa\mu}(\theta,\phi)$, and $\vec{\sigma}\cdot\vec{\ell}\chi_{\kappa\mu}(\theta,\phi) = -\frac{\kappa+1}{r}\chi_{\kappa\mu}(\theta,\phi)$ have been used, where $\vec{\sigma}\cdot\vec{\ell} = j^2 - \ell^2 - s^2$ and $j^2\chi_{\kappa\mu}(\theta,\phi) = j(j+1)\chi_{\kappa\mu}(\theta,\phi)$, $\ell^2\chi_{\kappa\mu}(\theta,\phi) = \ell(\ell+1)\chi_{\kappa\mu}(\theta,\phi)$, and $s^2\chi_{\kappa\mu}(\theta,\phi) = \frac{1}{2}(\frac{1}{2}+1)\chi_{\kappa\mu}(\theta,\phi)$. For states of ${}^1S_{1/2}$ symmetry, $\kappa = -1, \mu = 1/2$ (for angular momentum states j=1/2, $\ell = 0$)and similarly for the positron equation except that $\mu = -1/2$. Then the radial functions are represented by the trial forms $G_{-1}(r) = Ne^{-wr}$ and $F_{-1}(r) = -N\frac{\hbar cw}{E - V + mc^2}e^{-wr}$, where N is the normalization constant given by $N^2\{\int_0^\infty drr^2[G_{-1}^2(r) + F_{-1}^2(r)]\} = 1$. The same trial forms are used for both fermions, where w is a parameter which is varied to find a minimum in E. The latter step is carried out by making a first guess for E and inserting it wherever E occurs in a denominator. For a positive-energy state this guess is just $E = mc^2$, which is not updated since the positive-energy state occurs in the nonrelativistic regime, in which the binding energy is much less than mc^2 (Fig. 1). It is unlikely that the exponentially decaying trial forms with cusps at the origin will share any

energy with the unbound center of mass, which may be considered to be at rest. The error of course is in the absence of electron correlation in the relative motion, but we will tolerate this error in gratitude for a mathematically simple theory with which to investigate our assumption, which questions the orthodoxy of Dirac's hole interpretation of the negative-energy region of the spectrum, that Ps has a bound state in the negative-energy region.

For a negative-energy state E is guessed to lie close to $-mc^2$ and is updated until a minimum in E found which agrees with the guessed-at denominator energy as closely as possible. This match is not perfect owing to the approximate nature of the calculation, but an agreement of 99.6% is obtained (Fig. 4) for the guessed energy (-18732 au) and the energy calculated from Eq. (III-2) (-18659 au) where 1 au = 27.21 eV.

It is easy to understand the nature of the binding of the e^+e^- pair from analysis of Eq. (III-2). In the positive-energy region of the spectrum $F_{-1} \cong -N\frac{\hbar w}{mc}e^{-wr}$ such that $|F_{-1}| << G_{-1}$, which means that the first term on the right side of Eq. (III-2) is nearly equal to mc^2, the second term is negligible, and the lowest positive-energy bound state therefore lies just below mc^2 (Fig. 1). Notice that the Hartree model gives 80% of the known binding energy of 0.25 au. Figs. 2-3 give the family of wave functions for the values of the variational parameter, w, used in the calculation. These are presented in order to demonstrate the dominance of G_{-1} over F_{-1} in the positive-energy region.

On the other hand in the negative-energy region of the spectrum the denominator E-V+mc^2 neatly vanishes for $E \cong -mc^2$ such that $|F_{-1}| >> G_{-1}$, which means that the second term on the right side of Eq. (II-2) is nearly equal to -mc^2, the first term is negligible, and the lowest negative-energy bound state therefore lies just above -mc^2 (Fig. 4). (All bound states lie in the regime $-mc^2 < E < mc^2$.) The negative-energy bound state has not hitherto been reported due to the hole interpretation of one-body Dirac theory discussed at length above. Figs. 5-6 give the family of wave functions for the values of w used in the calculation. These are presented in order to demonstrate the dominance of F_{-1} over G_{-1} in the negative-energy region.

Remarkably the term in Eq. (III-2) proportional to κ, which is positive and therefore repulsive for $\kappa = -1$ and opposite phases for G_{-1} and F_{-1}, lifts the binding energy (E-mc^2) in the positive-energy region above its minimum value given by the sum of the second and third terms on the right side of Eq.(III-2) and is effectively the kinetic-energy contribution to the total energy, E. Conversely this kinetic-energy term and to a lesser extent the fourth and fifth terms on the right side of Eq. (III-2) lift the energy from a point below -mc^2, which lies in the negative-energy continuum, into the binding region above -mc^2. Remarkably a large kinetic energy in the negative-energy region makes binding possible by overcompensating for the second and third thirds term on the right side of Eq. (III-2), which are negative and therefore attractive and move E in the direction of less binding in the negative-energy region, in contrast to moving E in the direction of greater binding in the positive-energy region.

IV. Conclusion

In summary the Hartree model for two-body Dirac theory predicts that a bound state for Ps exists in the negative-energy region of the spectrum, such that annihilation and pair creation may be interpreted as ordinary two-photon emission and absorption respectively between the nominal ground state and the negative-energy state. This result shows that hole theory is not unique to the explanation of annihilation and pair creation and therefore lifts the ambiguity that Zitterbewegung [9,12] cannot be a real physical effect due to Dirac's hole interpretation of his one-body equation. It is hoped that this study will stimulate further work using more accurate two-body Dirac theory either to confirm or falsify the present Hartree-model result.

Ackowledgements. The author is grateful to T. Scott Carman for supporting this work. He is grateful to Professor John Knoblock of the University of Miami and to Dr. Ray Garrett of the University of Tennessee for seminal discussion. This work was performed under the auspices of the Lawrence Livermore National Security, LLC, (LLNS) under Contract No. DE-AC52-07NA27344.

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Figure Captions

Fig. 1. $E_+ - mc^2$ versus w for the lowest positive-energy state of Ps in the Hartree model.

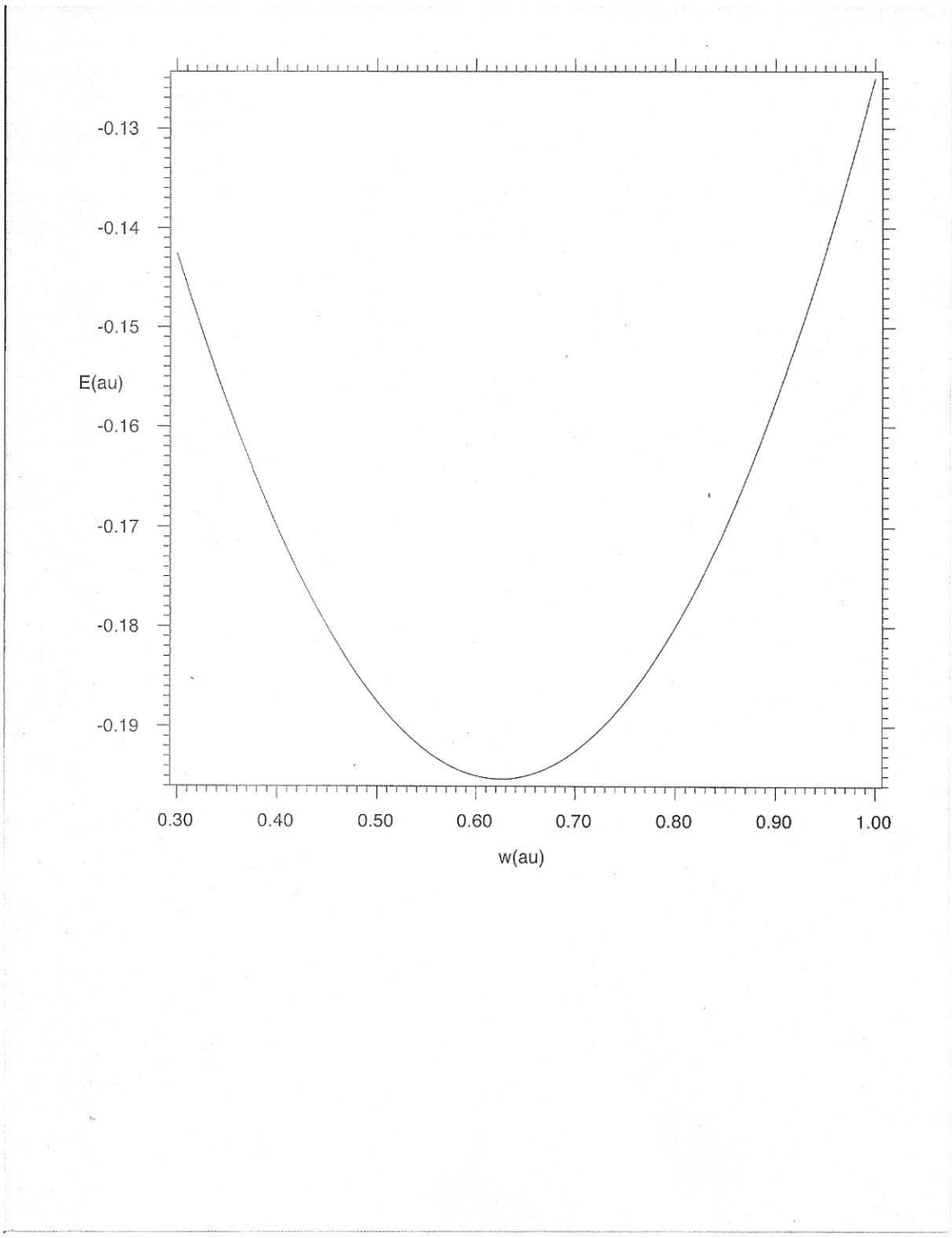
Fig. 2. $G_{-1}(r)$ versus r in the positive-energy region of the spectrum for the sweep of w values used in the calculation.

Fig. 3. $F_{-1}(r)$ versus r in the positive-energy region of the spectrum for the sweep of w values used in the calculation.

Fig. 4. E_- versus w for the lowest negative-energy bound state for Ps in the Hartree model.

Fig. 5. $G_{-1}(r)$ versus r in the negative-energy region of the spectrum for the sweep of w values used in the calculation.

Fig. 6. $F_{-1}(r)$ versus r in the negative-energy region of the spectrum for the sweep of w values used in the calculation.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w(au)

Fig. 1

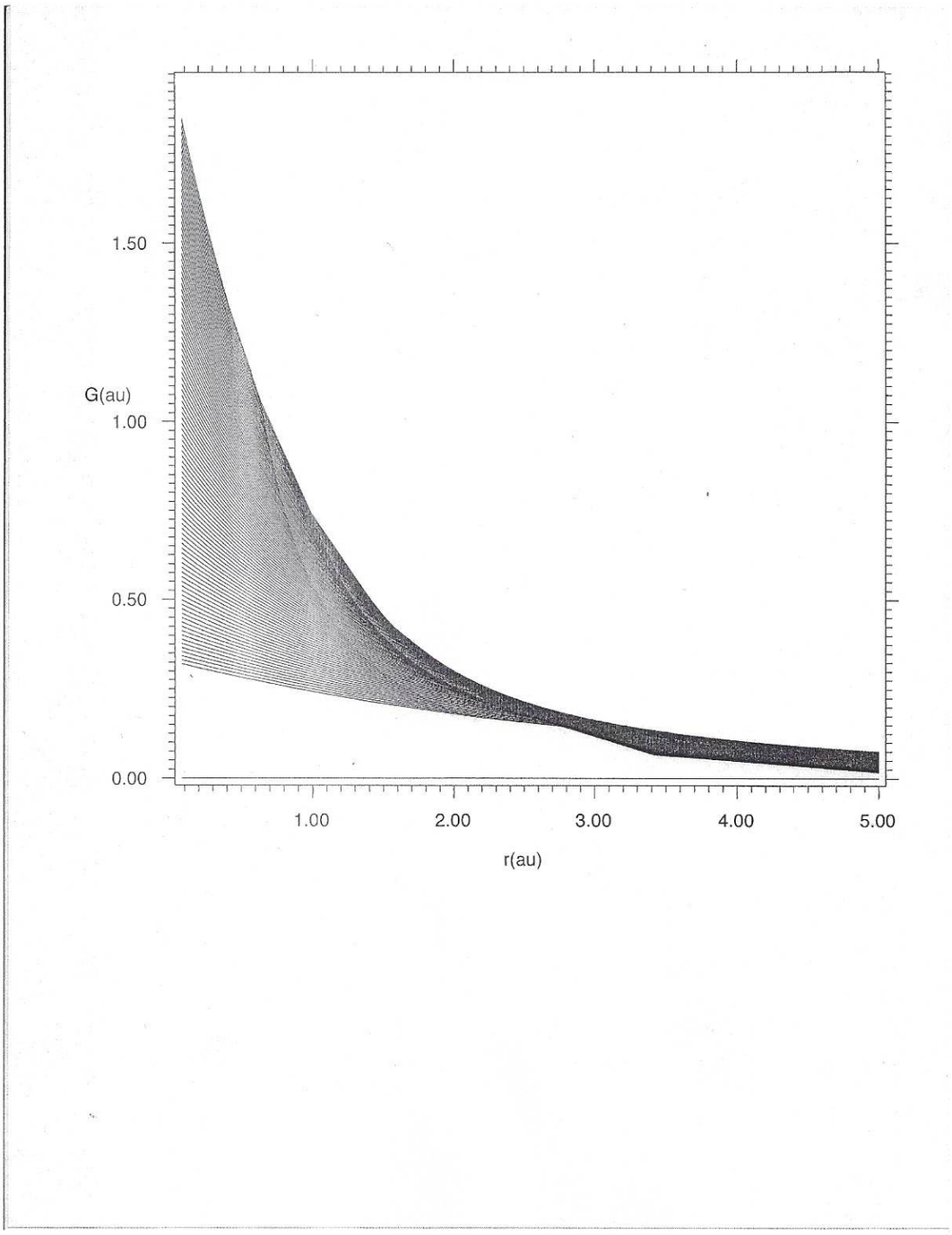

Fig. 2

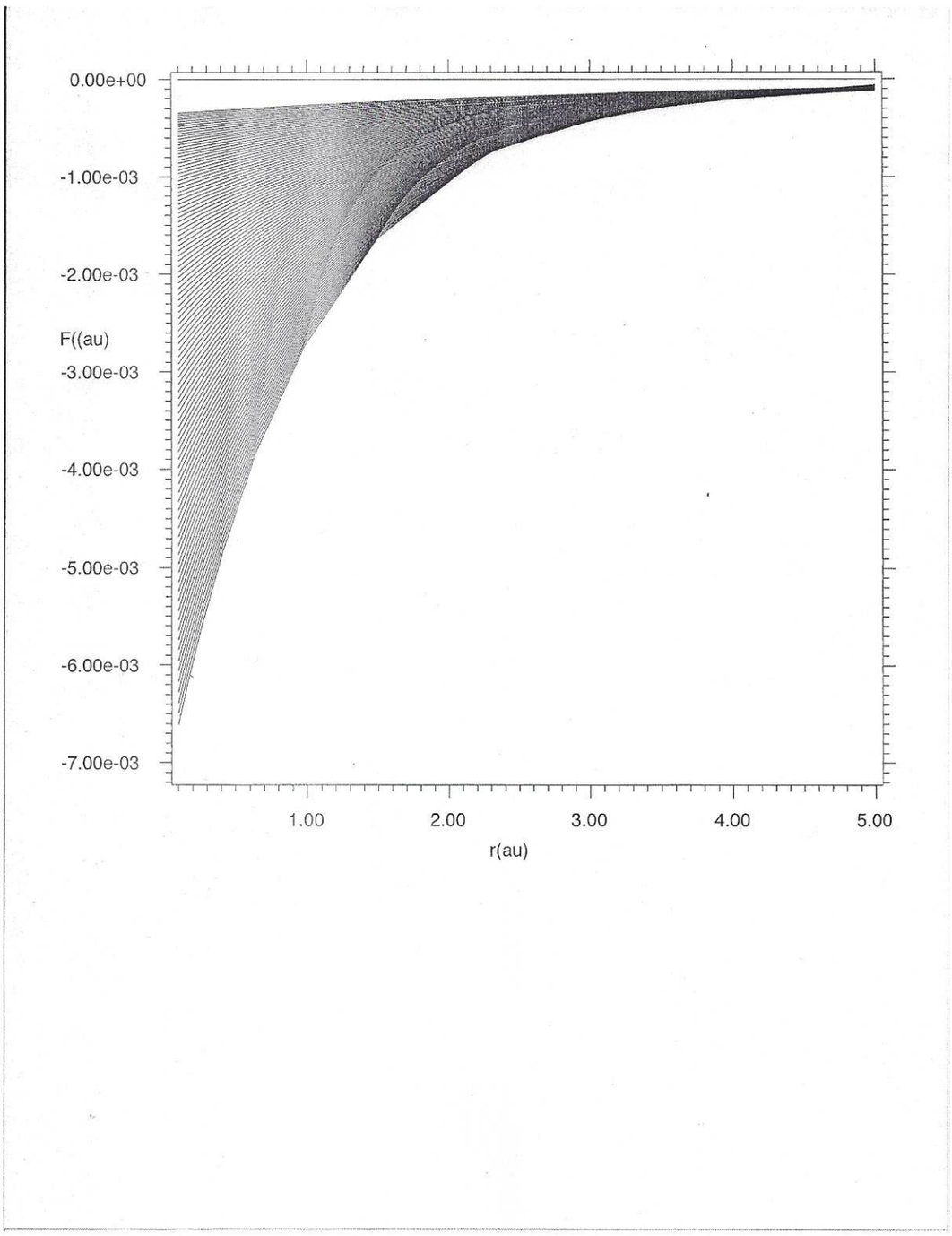
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Fig. 3

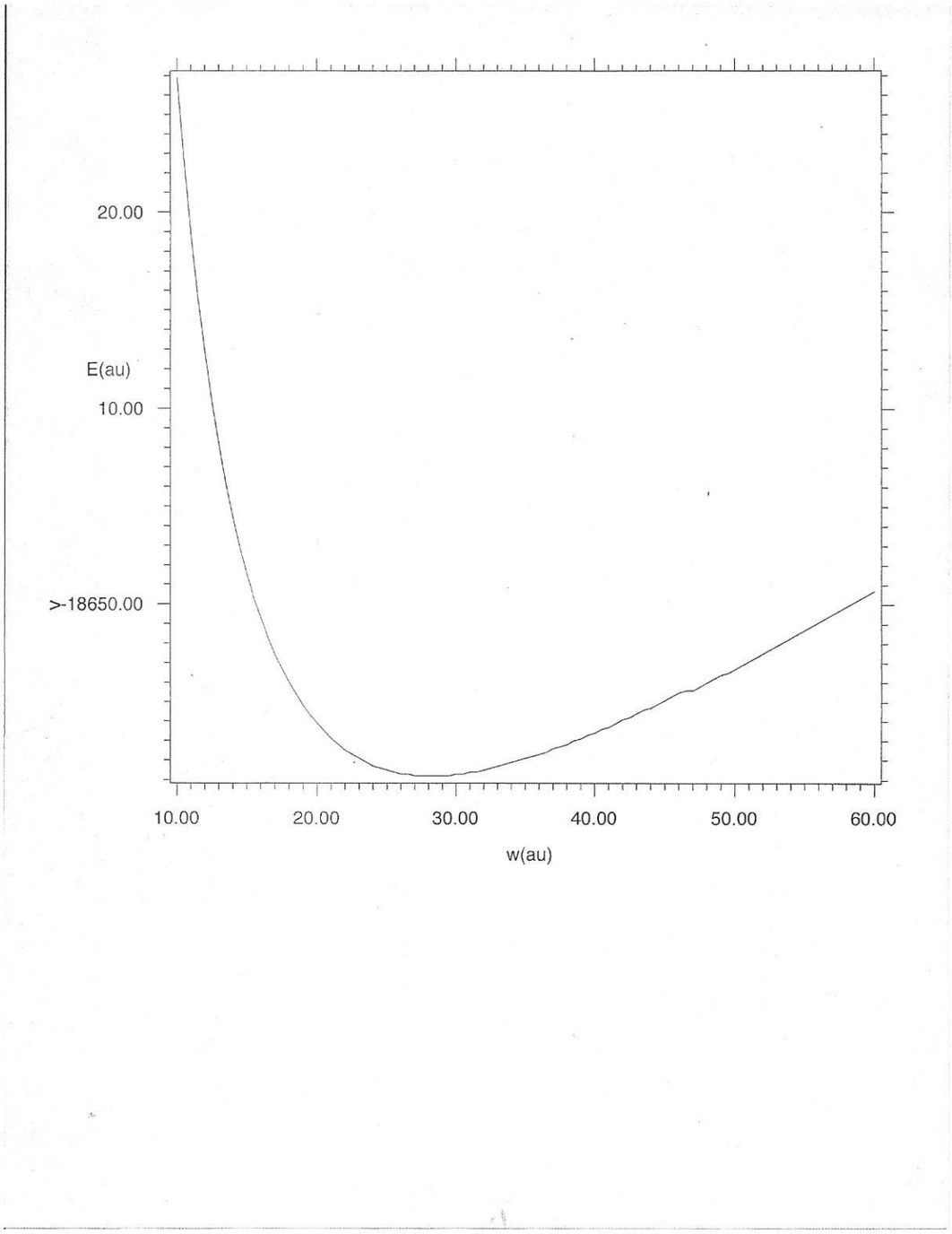

Fig. 4

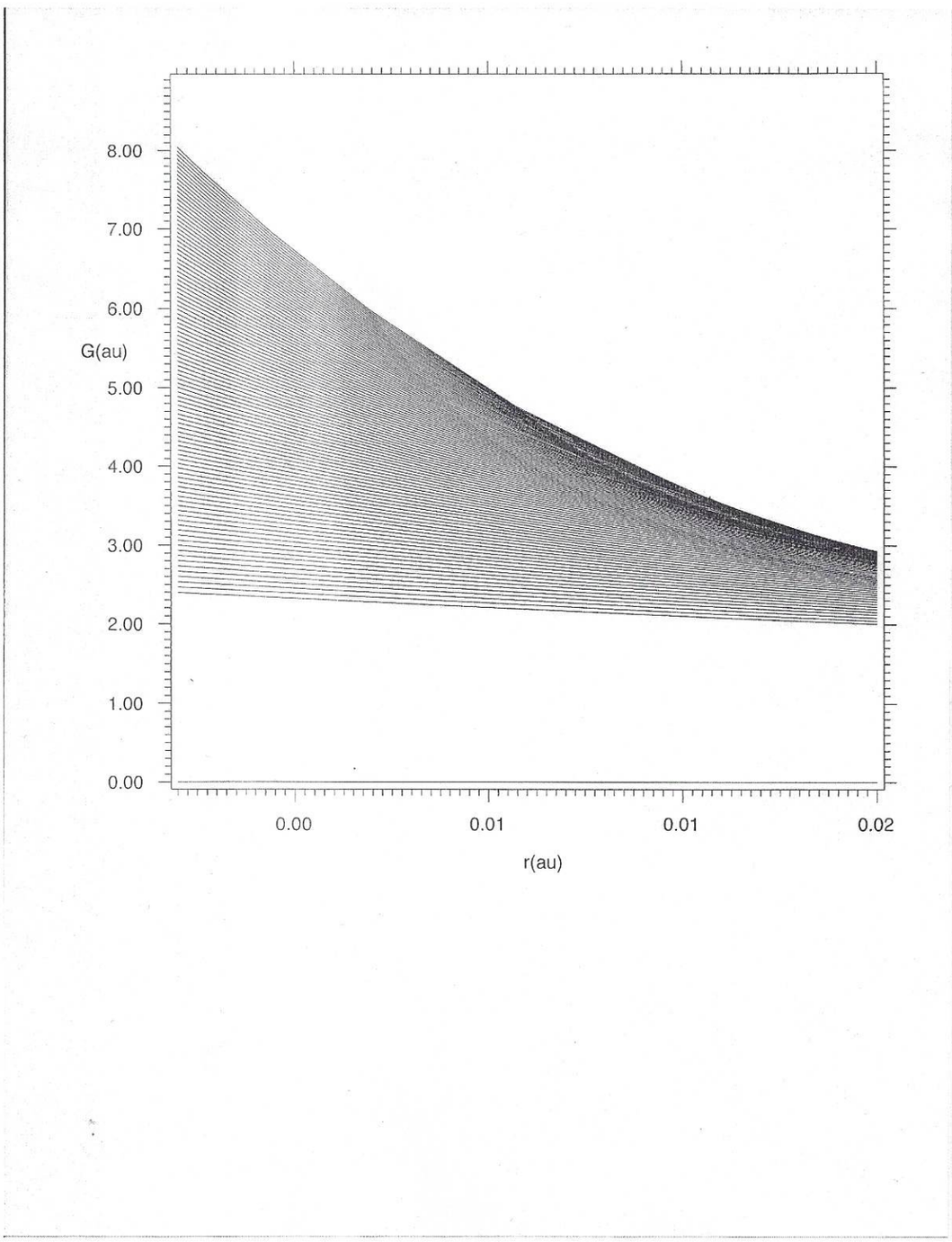

Fig. 5

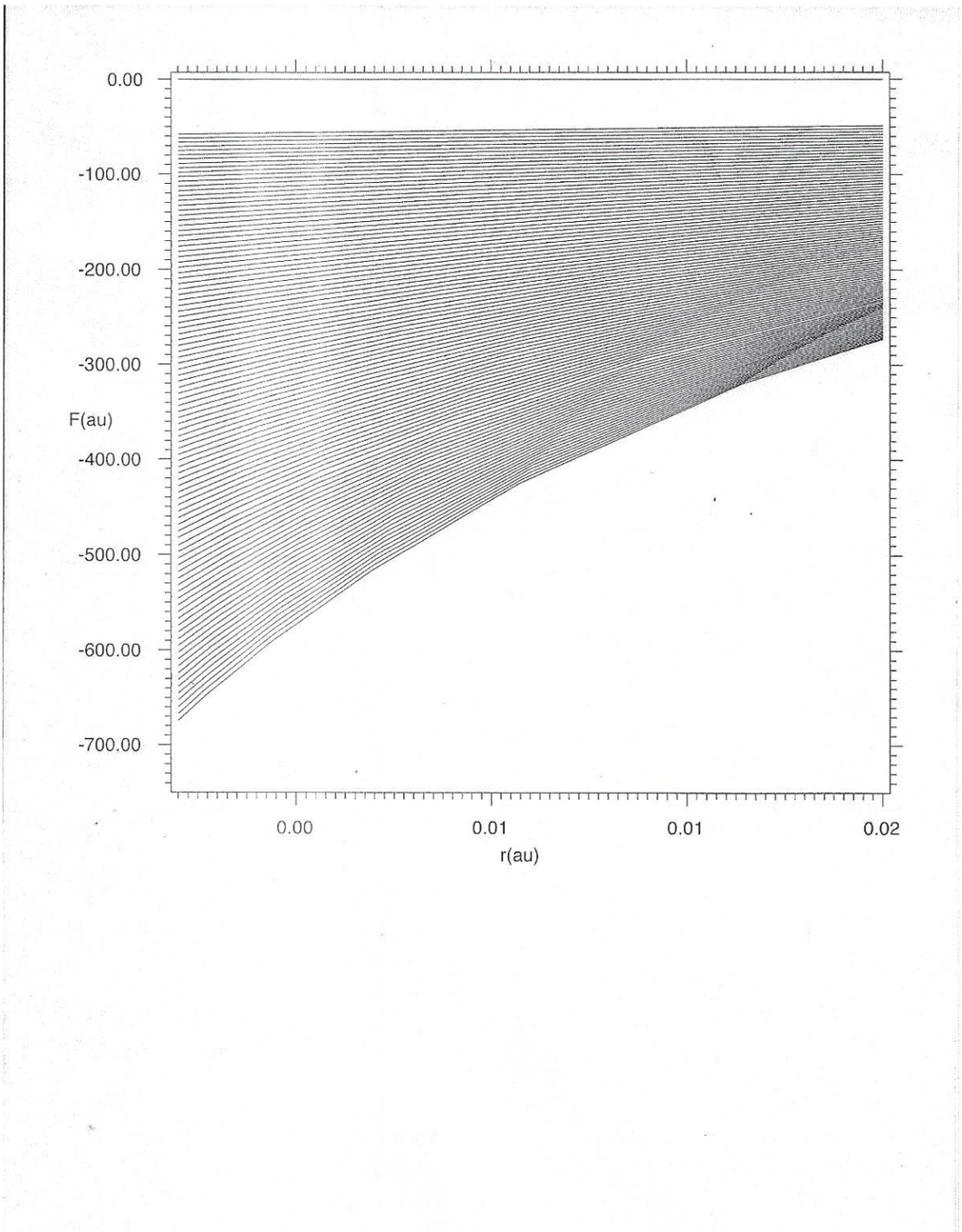
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Fig. 6