Original Research Article

Electromagnetic fields of eigenmodes in spherical resonators

ABSTRACT

In this article a physical theory of eigenmodes of electromagnetic resonators is presented. It is known, that Maxwell equations predict non-physical singular behavior of eigenmodes in spherical resonators. This shows that Maxwell theory is incomplete. For the improvement of the theory this problem is treated with the help of Maxwell-Einstein theory. Maxwell-Einstein equations take into account space-time curvature. Regular implementation of this approach permits to avoid the influence of singularity. Another result consists of that eigenmodes with large values of orbital angular moment are not observable. An analogy with CMB in the Universe is made.

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Keywords: resonator, eigenmode, singularity, cosmic microwave background

1. INTRODUCTION

Field in the electromagnetic resonators is usually described using the solutions of Maxwell's equations, which are superimposed by appropriate boundary conditions. These solutions looks like standing waves corresponding to the eigenmodes of resonators. If the resonator is exited by external sources it creates a field that can be represented as an series expansion in eigenmodes which form a complete orthogonal system. Below we are interested only in eigenmodes. In an empty cavity they are excited by radiation emitted by atoms of the cavity walls. Consider radiation of atoms located on one of the walls of resonator. For a qualitative analysis of the field of the resonator eigenmode we use the Huygens-Fresnel principle [1]. Suppose that each atom emits independently of the other, and thus the total radiation is a combination of waves of incoherent sources. Front of such a wave in no way corresponds to the shape of the cavity walls and the wave reaching the opposite wall, and reflected from it, will come to the original wall with random phase, which does not correspond to phase of the emitted wave. Thus, the reflected wave, having interacted with the original one, destroy it. This will not happen if the atoms radiate in phase. Then, the wave front shape corresponds to the shape of cavity wall, the reflected wave coming from the emitting panel having at each point the same phase shift, and if it is a multiple 2π[1], the resultant wave doesn't destroy and will comply with eigenmode of the resonator. In general, this situation is typical for the formation of eigenmodes for cavities of any shape. Of course, the condition for the survival of mode can't be considered as a reason for causing the wall atoms radiate coherently. The essential reason may be the synchronization atoms by eigenmode itself.

[1] For the eigenmodes with a sufficiently large number (spherical cavity)

The above picture is consistent with the definition of the eigenmode field using Maxwell's equations for rectangular resonators, when their solutions have no singularities and have simple interpretation. For resonators of spherical shape solutions of Maxwell equations have a singularity at $r = 0$, what requires assumptions about the nonphysical infinite energy density at the origin, which is located in the center of the cavity. When one tries to give a physically meaningful interpretation of the eigenmodes of spherical resonators this fact must be taken into account and requires going beyond the Maxwell theory. First physically reasonable solution to this problem was proposed in the paper [2], devoted to the definition of the metric of space-time, curves by spherical electromagnetic waves (SEMW). This requires along with Maxwell's equations also use the Einstein equations for the Riemann tensor, which describe the curvature of space-time, and which right side contains energy-momentum tensor of SEMW. This is justified, at least by two reasons:

1. The metric tensor of the problem [2, 3] contains a component that is independent of the amplitude of the electric wave and significant at distances of the order of the wavelength.
2. In general solutions of Einstein's equations have singularities, which can prove as an example of specific solutions (Schwarzschild metric), and with the help of the theorems on the global structure of space-time [4].

An attempts was made to interpret the singular solutions using the Maxwell equations alone (or methods of geometrical optics) for the fields in the cavities or open optical systems, focusing the incident field at the point (focus), but did not give conclusive results [5][2]. These failures can be considered as a third reason justifying the use Maxwell-Einstein equations to solve the problem.

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2. ANGULAR DISTRIBUTION OF THE EIGENMODES OF SPHERICAL RESONATORS

Solutions of the Maxwell equation which was used in [2][3] obey degeneration, connecting with arbitrariness of z – axis' direction of co-ordinate system. If direction of z – axis is fixed the initial spherical symmetry of problem is lowered.

In quantum mechanics recovery of breaking symmetry is due to so-called zero modes [7]. In our problem all directions of z – axis are equivalent: all solutions corresponding to its different directions are possible and have the same energy. In order to eliminate zero modes, one must explicitly take into account the transitions between degenerate states. For the simplicity one can do this in quantum description. A simplification of problem is connected with fact that angular behavior of photon wave function is just the same as for classical SEMW. Let us calculate the probability of transition from the state with orbital quantum number l, which angular behavior is described by $P_l(\cos(\theta))$ in co-ordinate system with given axis z, to the state with the same quantum number in co-ordinate system with axis z' deviating from z on angle $\Delta\theta$. In this latter co-ordinate system angular behavior of wave function describes as $P_l(\cos(\theta+\Delta\theta))$[4]. The amplitude of the interested probability is equal to the projection of the shifted state $P_l(\cos(\theta+\Delta\theta))$ on the unshifted one $P_l(\cos(\theta))$ (both normalized):

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[2] In [5] a notion of an effective sources for divergent SEMW so on as sinks for convergent ones are introduced.

[3] So as all similar solutions, which can be found in scientific literature (see [6], for example). As a consequence, field distribution in spheroidal electromagnetic resonator has axial symmetry.

[4] P_l are Legendre polinomials, P_l^k- are associated Legendre polinomials.

$$I_l(\mathit{\Delta}\theta) = \frac{2l+2}{2}\int_0^{\pi} P_l(\cos\theta)P_l(\cos(\theta+\mathit{\Delta}\theta)\sin\theta d\theta \qquad (1)$$

The amplitude of transition probability $I_k(\Delta\theta)$ one can find with the help of addition theorem for spherical functions [8]:

$$P_l(\cos(\theta+\Delta\theta)) = P_l(\cos(\theta))P_l(\cos(\Delta\theta)) + 2\sum_{k=1}^{l}\frac{(l-k)!}{(l+k)!}P_l^k(\cos(\theta))P_l^k(\cos(\Delta\theta)) \qquad (2)$$

Due to orthogonality of associated Legendre polinomials [8], we receive:

$$I_l(\mathit{\Delta}\theta) = P_l(\cos\mathit{\Delta}\theta) + 2\sum_{k=1}^{[l/2]}\frac{(-1)^k l!}{(l+2k)!}P_l^{2k}(\cos\mathit{\Delta}\theta) \qquad (3)$$

Symbol $[x]$ means integer value of x. We give below expressions for the first five values $I_l(\Delta\theta)$:

$$\begin{aligned}
&I_0(\mathit{\Delta}\theta) = 1,\\
&I_1(\mathit{\Delta}\theta) = P_1(\cos\mathit{\Delta}\theta),\\
&I_2(\mathit{\Delta}\theta) = P_2(\cos\mathit{\Delta}\theta) - \frac{1}{6}P_2^2(\cos\mathit{\Delta}\theta),\\
&I_3(\mathit{\Delta}\theta) = P_3(\cos\mathit{\Delta}\theta) - \frac{1}{10}P_3^2(\cos\mathit{\Delta}\theta),\\
&I_4(\mathit{\Delta}\theta) = P_1(\cos\mathit{\Delta}\theta) - \frac{1}{15}P_4^2(\cos\mathit{\Delta}\theta) + \frac{1}{840}P_4^4(\cos\mathit{\Delta}\theta)
\end{aligned} \qquad (4)$$

Interested probabilities look as $w_l = (I_l(\Delta\theta))^2$

Fig.1 represents results of calculation $w_l(\Delta\theta)$ for different values of l:

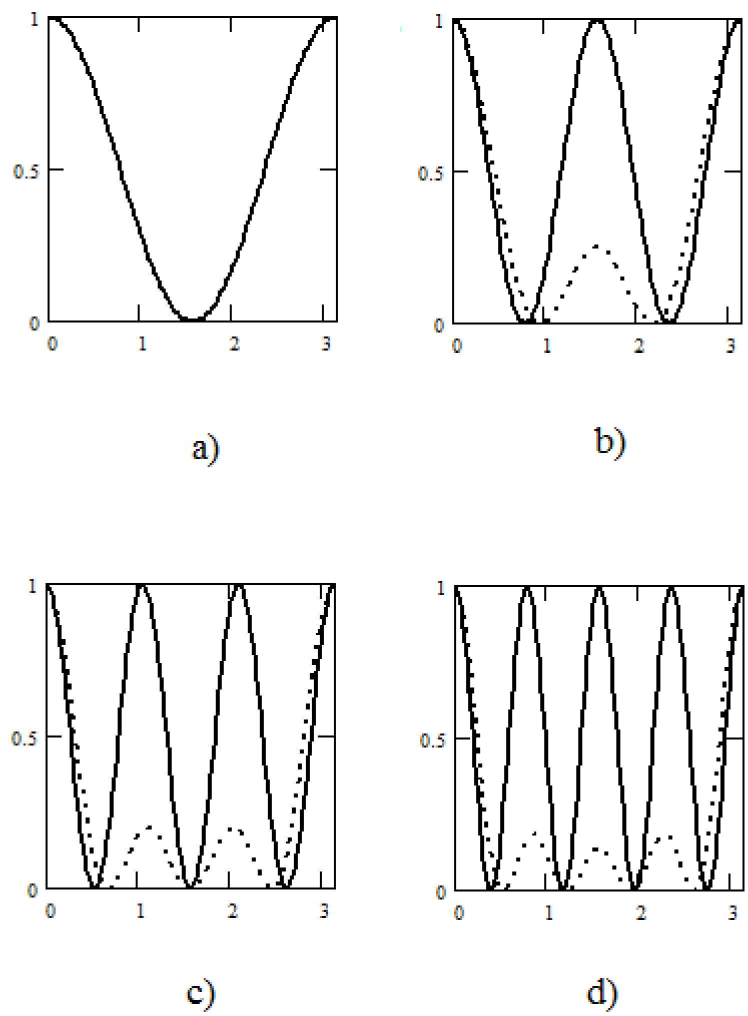

Fig. 1. Plot of $w_l(\Delta\theta)$

Ordinata: points – $(P_l(cos(\theta)))^2$, solid curve – $w_l(\Delta\theta)$; Abscissa: angles θ and $\Delta\theta$ from 0 to π;

a) $l = 1$ (curves coincide), b) $l = 2$, c) $l = 3$, d) $l = 4$.

These results show that angular region $\Delta\theta_c$, where fraction of “shifted” harmonic $P_l(cos(\theta+\Delta\theta))$ in the “basic” one $P_l(cos(\theta))$ is significant, is comparable with scale θ_c of angular dependence of $P_l(cos(\theta))$, which has order of value $1/l$. Mathematically it is due to the interference of different terms in (3) and (4). Physically this can be assigned to effect of zero modes, because both abovementioned harmonics have the same energy. Of course, this effect vanishes when direction of z axis is fixed physically, for instance, with the help of external field.

Recall that the field of electrical oscillations in a spherical cavity is defined by the function U, which has the form [6]

$$U = A\Psi_l(kr)P_l^m(\cos\theta)\begin{Bmatrix}\cos m\varphi\\ \sin m\varphi\end{Bmatrix} \qquad (5)$$

A – is a constant, P_l^m – associated Legendre polynomial, *r, θ, φ* – spherical coordinates, $\Psi_l(kr)$ –radial part of the field, *k* – wavenumber. Equations for *U* are given in [6]. We have already mentioned that the expression (5) is valid only for the modes excited by an external source (antenna), which specifies the direction of the *OZ* axis of coordinate system. Question about the arbitrariness of the choice of direction of the axis *OZ* is also discussed in [6], but the answer seems unconvincing.

In the spirit of this approach the correct expression for the eigenmode field *U* of a spherical cavity must take into account the degeneracy of the directions the axis *OZ*. Simplify the problem by putting $m = 0$. This means that we fix a plane in which lies the axis *OZ* so it is perpendicular to the kinetic moment of the wave. As mentioned above, all directions $\theta_{0n} = \pi n/l,\ 0 \le n \le l - 1$ in this plane, measured from some arbitrary reference direction $\theta_{00} = 0$, may be taken on the same ground as the orientation of axis *OZ.*

Desired expression for *U* must be of the form (in the general case $m \ne 0$)

$$U' = B\Psi_l(kr)\sum_{n=0}^{l-1} P_l(\cos(\theta - \theta_{0n}))\begin{Bmatrix}\cos m\varphi\\ \sin m\varphi\end{Bmatrix} \qquad (6)$$

B – is a constant defined as *A* in (5) by the normalization condition. Due to linearity of Maxwell's equations (6) is a mode of a spherical cavity, but in contrast to (5), it corresponds for the maximum degree of symmetry of the problem.

Expression (6) was used in [2, 3] when recording energy-momentum tensor of SEMW before averaging it over the angle *θ*. Such a procedure is always applied when considering the free-oriented systems [9].

3. ELECTROMAGNETIC FIELDS OF THE EIGENMODES OF SPHERICAL RESONATORS[5]

Eigenmodes In the spherical cavity also are excited by atoms of wall which are synchronized by radiated wave. When one traditionally considers the spherical cavity eigenmodes within Maxwell's theory he will receive for the radial parts of the complex field amplitudes well-known expressions of the form of standing waves ~ $J_{n+1/2}(kr)/(kr)^{3/2}e^{i\omega t}$, containing the above-mention singularity [6] (*J* - Bessel function, *r* - radial coordinate, *k* - wave number, *ω* - the angular frequency, *n* - integer). Physically reasonable to present them as the sum of a convergent (~ $e^{i(kr+\pi n/2+\omega t)}$) and divergent (~ $e^{i(-kr+\pi n/2+\omega t)}$) waves[6]. The radiation of the atoms of walls excites convergent wave, which converges to the point $r = 0$, passes it some way, then is transformed into a divergent one, reaches the walls of the cavity and, in the case of a phase shift multiple to 2π creates a stable eigenmode. The latter condition determines the mode spectrum, i.e. a set of allowed values $\omega=\omega_n$[7]. This is reminiscent of the argument given above for the rectangular cavity. There is, however, a subtle place associated with the passage of the convergent wave the point $r = 0$. As was shown in [3], convergent wave is

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[5] Some subsequent material were published in summary form at the conference Saratov Fall Meeting, SFM'13 as Internet report [10]

[6] Given expressions are valid for $kr >> 1$.

[7] In electromagnetic theory eigenmode spectrum is obtained from boundary condition on the wall of the resonator at $r = R$, *R* – is the radius of resonator which leads to an equation $J_{n+1/2}(kR)=0.$

partially captured by the curvature of the metric at $r = 0$ in the domain which size is of the order of the wavelength λ and can't conventionally, i.e. classically be transformed into divergent one. For this to happen, it is necessary to involve solutions of the M-E equations of another, non-wave type, the existence of which is proved in [2][8]. This remines the tunneling process in quantum mechanics: a convergent electromagnetic wave is transformed into an instanton, and from it - in the divergent wave. This process occurs with probability $w \sim exp(-\Lambda_0/\hbar)$, where $\hbar = h/2\pi$, h – is Plank constant, Λ_0 - finite pseudo-euclidean action of the instanton [2, 11]. Thus, each eigenmode of spherical cavity has a probability $w = w(\omega)$[9]. Electromagnetic field of the instanton and the magnitude Λ_0 were calculated in [2] and [11]. The results of both papers agree qualitatively. In [2], the action of the instanton Λ_0 was determined from the equations for the electromagnetic field produced by a variation of the action S of the field on the independent components of the field tensor F_{lk}. In [11] action Λ was recorded taking into account ties imposed on components F_{lk}, arising out of the field equations, and then the variation $\delta\Lambda$ was calculated and action Λ_0 was determined from the condition $\delta\Lambda = 0$.

According to the results of [11] the instanton field is exponentially small at the vicinity of $r = 0$, that is corresponding to the nature of the tunnelling, and solves the problem of singularity of field of spherical electromagnetic wave at the point $r = 0$, although the metric is singular at this point.

4. ISOTROPISATION OF EIGENMODES IN SPHERICAL RESONATORS

Space-time metric, curved by the presence of a SEMW is found in [3] and looks as follows

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\begin{aligned}
&ds^2 = e^{-\alpha}c^2dt^2 - e^{\alpha}dr^2 - r^2(d\theta^2 + \sin^2\theta \cdot d\varphi^2) \\
&e^{-\alpha} = g_{00} = 1 - \frac{r_c}{r} + \left(\frac{r_s}{r}\right)^2 \\
&r_c = \frac{l(l+1)c}{\omega}, r_s^2 = \frac{K}{2c^4}|G|^2
\end{aligned}
\tag{7}
$$

G - the amplitude of the electromagnetic wave, l - an integer specifying the orbital angular momentum of the SEMW, K - the gravitational constant. Eigenmodes of the spherical cavity, as shown in [3], can be divided into scattered by curvature of metrics and captured by it. Scattered modes in terms of geometrical optics are associated with rays, which are corresponding to the areas of the front of the SEMW satisfying conditions $\theta < \theta_*$ or $\pi > \theta > \pi - \theta_*$, where θ- polar angle, and

$$
Sin\theta_* = \frac{r_c}{\rho_*}\frac{m}{l(l+1)}
\tag{8}
$$

ρ_* - is impact distance of the ray, at which capture takes place for the first time, m – integer which defines projection of orbital kinetic moment on axis OZ, $-l < m < l$ [3]. All other modes

[8] In [2] they were called as instanton-like solutions

[9] Coincidence of mode's frequency distribution with Plank one permits to connect instanton parameters with temperature of equilibrium radiation in cavity.

are captured by the curvature of the metric. Here we consider the scattered modes and clarify their role in shaping the field of eigenfields of the spherical electromagnetic resonators.

As is known from electromagnetic theory, electromagnetic fields in spherical resonators have axial symmetry [6]. This is due to the fixed direction of axis *OZ*, from which the angle θ is measured. This is true for forced oscillations in resonators excited by an external source, such as an antenna, which sets the preferred direction. However, for eigenmodes none of the preferred directions as the orientation of the axis *OZ* among others can be selected. The only thing that can be observed in the experiment - is the angular distribution of the eigenmodes. It doesn't permit to determine unequivocally the direction of axis *OZ*. For example, for $l = 1$ (dipole mode) directions corresponding to $\theta=0$ and $\theta=\pi$ are equivalent and both may be selected as the orientation of axis *OZ*. For $l > 1$ the situation becomes even more ambiguous: all directions $\theta=m\pi/l$ are equivalent (see Fig. 1). The situation is exacerbated when one considers modes scattered by curved metric. To summarize, for $l >> 1$ any direction can be selected with an equal basis as the axis *OZ*, because the angular distribution of the higher modes becomes completely isotropic and gives no basis for choosing a particular direction as the axis *OZ*. This can be illustrated with the following considerations. Mode of the order l has an angular distribution (in the angle θ), which is characterized by the maximums of width $1/l$. Near each maximum scattered modes are concentrated, the maximum deviation angle[10] of which is determine by the formula $\delta\vartheta \approx 2r_c / \rho_*$ (if one neglects the amplitude of SEMW[11]) [3]. For the impact distance ρ_*, with which the capture begins, one can take a value $\rho_* = \sqrt{27}r_c / 2$ which is corresponding to SEMW of small amplitude [3, 15]. Then, one receives $\delta\vartheta_{\max} = 4/\sqrt{27} \approx 0.77$. Overlapping of neighboring peaks occur when the inequality $1/l + \delta\vartheta_{\max} \geq \pi / l$ will be valid, what takes place for $l \geq 3$. Thus, the observation of non-uniform angular distribution of the eigenfields of the spherical resonator is possible only for small $l = 1, 2$. This corresponds to values of j, defining full kinetic moment of SEMW $j = l \pm 1 = 1, 2, 3$ (value $j = 0$ is forbidden). For the eigenfields of higher order electromagnetic fields are isotropic because peaks of angular distributions of them are overlapping. Recall that we are talking about the amplitude of oscillation, it phase retains the dependence on the azimuthal angle φ.

5. APPLICATION TO COSMIC MICROWAVE BACKGROUND

It is interesting that these results are applicable in cosmology. Indeed, the cosmic microwave background (CMB) shows features characteristic of eigenmodes of spherical resonators: a high degree of isotropy and Planck frequency distribution [12]. To reinforce the analogy, we note two facts. First, there is a model of the universe[12], representing it as a spherical cavity with a radius increasing with time [13, 14]. The role of the walls of that cavity plays so-called surface of last scattering. CMB radiation in this model is represented as a standing electromagnetic waves - the eigenmodes of the cavity. This model predicts the correct dependence of the radiation frequency on the radius of the Universe[13] [13]. It should be noted that, despite the different nature of the sources of the eigenmodes of the resonator and the relict radiation of the universe, the analogy between them is permissible, because the received radiation is likely not the primary born as a result of annihilation processes in

[10] Defined by the angle of deflection of the ray corresponding to a small part of the front of the SEMW
[11] What can be done for the entire observable universe.
[12] Closed model of the universe
[13] Strictly speaking, in these arguments the role of the radius of the universe should play radius of the sphere of last scattering.

lepton-baryon plasma that filled the universe immediately after Big Bang. Between the birth of the primary photons of the CMB and their detection by devices considerable time has passed, during which in the “universe – resonator” could finish transients formed the standing waves, taken as a relict by devices.

Secondly, it follows from the experimental data, the CMB radiation in the long wave limit can be described as classical electromagnetic waves. According to generally accepted ideas CMB – is a photon gas which is formed in the Big Bang and is currently in thermal equilibrium at temperature ~ $2{,}7^o\ K$ [12]. This gas fills the universe, which is described by one of the cosmological models, which are based on Einstein's equations. CMB is observed in the range from *0.33 Sm* to *73.5 Sm* [12]. In long wave diapason of the CMB quantum numbers of photonic levels occupation $N_k = \langle E^2\rangle c^3/ \hbar\omega^4 \gg 1$ [15], $\langle E^2\rangle$ - the average energy density of the microwave radiation, which is equal to $4\cdot 10^{-20}\ J/Sm^3$ [15]. This allows one to use classical equations for its description. Shortwave portion of CMB radiation for which $N_k \ll 1$, by contrast, allows one to apply the concepts of geometrical optics.

6. CONCLUSIONS

Results of this paper have revealed the role of the curvature of space-time metric due to the electromagnetic field of the waves in the cavity on the nature of the eigenmodes in the cavity. It gives the possibility to overcome the shortcomings of a purely mathematical approach to solving the problem of the study the eigenmodes in a spherical electromagnetic resonator based on the solution of Maxwell's equations and develop a physical theory of eigenmodes in spherical cavity.

The study of the electromagnetic eigenmodes of a spherical cavity using Einstein-Maxwell equations leads to elimination of the non-physical singularity of the wave field in the center of the cavity. This was made possible thanks to the fact that the process of transformation of a converging spherical electromagnetic waves (SEMW) in diverging one occurs through instanton which is a non-wave solution of Einstein-Maxwell equations [11]. Instanton parameters can be expressed in terms of the temperature of the radiation in the resonator. Another consequence of the physical approach is that every eigenmode has a probability which defines its existence.

Capture rays corresponding to SEMW with large values of the orbital angular momentum l leads to that only modes corresponding to small values of $l=1,2$ can be observed. Eigenmodes with large values will not be observed.

This result relating to cosmology, give reason to assume that the observed anisotropy of CMB associated with harmonics with low values of the orbital angular momentum and attributed to Intergalactic movements may actually be the property of the CMB caused by the influence of the curvature of space-time metric, created by them.

Another result concerns the focusing of rays in the lens system. We have already mentioned about trying to solve this problem using fictitious sinks and sources [5]. Consideration of this problem in the curved space-time allows us to give another solution. Following analogy with solutions of Einstein's equations, near the space-time singularity is permissible. It is known that the minimum area of a sphere of radius r in the space-time possessing a Schwarzschild metric is $S_{min} = 4\pi r_g^2$, r_g – gravitational radius [17]. In our problem with the metric (7), the role of gravitational radius plays the r_c[14] . Instanton allows sphere (spherical front of the SEMW)

[14] At distances $r \sim r_c$ last term in (7) can be neglected, so there is a complete analogy with the Schwarzschild problem [15].

after reaching the minimum area to expand in the same region *I*, from which it began its convergence, but not in the unphysical region *I′* [17][15].

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[15] The latter is a figure of speech [17], because there is no time- like geodesic going from *I* to *I′*.