Original Research Article

DYNAMIC BUCKLING LOAD OF AN IMPERFECT VISCOUSLY DAMPED SPHERICAL CAP STRESSED BY A STEP LOAD

ABSTRACT

This paper determines the dynamic buckling load of a lightly and viscously damped imperfect spherical cap with a step load. The spherical cap is discretized into a pre-buckling symmetric mode and a buckling mode that consists of axisymmetric and non-axisymmetric buckling modes. The imperfection is taken at the shape of the buckling mode. The inherent problem contains a small parameter which necessitated the adoption of regular perturbation procedures, using asymptotic technique. The general result is designed to display the contributions of each of the terms in the governing differential equations. We deduce the results for the respective special cases where the axisymmetric imperfection parameter, namely $\bar{\xi}_1$, and the non-axisymmetric imperfection parameter $\bar{\xi}_2$, are zeros. We also determine the effects of each of the non-linear terms as well as the effects of the coupling term.

Keywords: Spherical cap, step load, dynamic buckling, imperfection parameter.

1. INTRODUCTION

The analysis of the dynamic buckling load of elastic structures was primarily enunciated by $[1-3]$.Other pertinent investigations include $[4-14]$, among others. However, a cursory appraisal of all the investigations to date reveals that the phenomenon of damping has been given very little or no attention at all in the dynamic buckling process. We are of the strong opinion that since dynamic buckling process is a time dependent process, the effect of damping, no matter how slight, should not be overlooked. In this investigation, the presence of a small light viscous damping is therefore assumed.

The dynamic buckling loads of imperfection-sensitive structures from perturbation procedure were analyzed by $[15]$ in which he invoked multiple-timing procedure and made use of Mathieu-type instability. The present study is an extension of $[15]$ to the case where a small light viscous damping is present. We however avoid Danielson's method, for, as noted by $[3]$, Mathieu-type instability is always associated with many cycles of oscillations as opposed to just one shot of oscillation that triggers off dynamic buckling. There are five sections in this paper. Section two examines the dynamic buckling load of an imperfect viscously damped spherical cap stressed by a step load. Section three introduces the viscous damping to Danielson's results. Section four considers the analysis of results while section five ends this work with a conclusion.

2. THE DYNAMIC BUCKLING LOAD

Danielson, had, for simplicity, assumed that the normal displacement $W(x,y,T)$ of the spherical cap was given as

$$W(x,y,T) = \xi_0(T)W_0(x,y) + \xi_1(T)W_1(x,y) + \xi_2(T)W_2(x,y) \qquad (1)$$

where $W_0(x,y)$ is the pre-buckling mode and $W_1(x,y), W_2(x,y)$ are the axisymmetric and non-axisymmetiic modes respectively. $\xi_0(T), \xi_1(T)$ and $\xi_2(T)$ are the respective time dependent amplitudes of the associated modes. Imperfection $\bar{W}$ was introduced as $\bar{W} = \bar{\xi}_1 W_1 + \bar{\xi}_2 W_2$

(2)

where W_1, W_2 still have meanings as before and $\bar{\xi}_1, \bar{\xi}_2$ are the imperfect amplitudes assumed to be small relative to unity. On assuming suitable forms for W_0, W_1, W_2 and substituting same into the compatibility and dynamic equilibrium equations and simplifying, using his assumptions, Danielson obtained the following coupled differential equations for step loading

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{dT^2} + \xi_0 = \lambda f(T)$$

(3) $$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{dT^2} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi_2^2 = \bar{\xi}_1\,\xi_0 \qquad (4)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{dT^2} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_2\,\xi_0$$

(5) $\xi_i(0) = \xi_i'(0) = 0; i = 1, 2.$

Here, $f(T)$ is the loading history which in our investigation, (as in Danielson's case), is the step load characterized by

$$f(T) = \begin{cases} 1, T>0 \\ 0, T<0 \end{cases} \qquad (6)$$

and, λ ,is the load parameter, considered to be non-dimensionalized and satisfies the inequality $0 < \lambda < 1$.

As in (3)-(5), we note that $\omega_i; i = 0,1,2$ are the circular frequencies of the associated modes ξ_0, ξ_1 and ξ_2 respectively while k_2 and k_2 are constants considered positive

3. THE USE OF VISCOUS DAMPING IN DANIELSON'S RESULTS

In our guest for solution, we are to determine a particular value of λ , called the dynamic buckling load represented by λ_D and which satisfies the inequality $0 < \lambda_D < 1$. Relatively, recent investigations that have tended to incorporate damping include $[17-20]$.For simplicity of analysis, we assume the existence of damping on the buckling modes. Since this damping must be only proportional to the velocity, we add the terms $c_1 \frac{d\xi_1}{dT}$ and $c_2 \frac{d\xi_2}{dT}$ to (4) and (5) respectively and the formulation now becomes

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{dT^2} + \xi_0 = \lambda f(T)$$

(7) $$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{dT^2} + c_1\frac{d\xi_1}{dT} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi_2^2 = \bar{\xi}_1 \xi_0$$

(8) $$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{dT^2} + c_2\frac{d\xi_2}{dT}\xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_2 \xi_0$$

(9)

where $c_i, i = 1, 2$ are the damping constants and which satisfy the inequality $0 < c_i < 1$.

Using $f(T) = 1$ and substituting (6) into (7) we have

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{dT^2} + \xi_0 = \lambda \tag{10}$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{dT^2} + c_1\frac{d\xi_1}{dT} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi_2^2 = \bar{\xi}_1 \xi_0 \tag{11}$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{dT^2} + c_2\frac{d\xi_2}{dT} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_2 \xi_0 \tag{12}$$

Now using,

$t = \omega_0 T$,

so that

$$\frac{d(\ )}{dT} = \omega_0 \frac{d(\ )}{dt}, \frac{d^2(\ )}{dT^2} = \omega_o^2 \frac{d^2(\ )}{dt^2},$$

Then $(10)-(12)$ become

$$\frac{d^2\xi_0}{dt^2} + \xi_0 = \lambda \qquad (13)$$

$$\frac{d^2\xi_1}{dt^2} + \left[\frac{c_1\omega_0\omega_1^2}{\omega_0^2}\right]\frac{d\xi_1}{dt} + \left[\frac{\omega_1}{\omega_0}\right]^2 \xi_1(1-\xi_0) - \left[\frac{\omega_1}{\omega_0}\right]^2 k_1\xi_1^2 + \left[\frac{\omega_1}{\omega_0}\right]^2 k_2\xi_2^2 = \left[\frac{\omega_1}{\omega_0}\right]^2 \bar{\xi}_1\xi_0 \qquad (14)$$

$$\frac{d^2\xi_2}{dt^2} + \left[\frac{c_2\omega_0\omega_2^2}{\omega_0^2}\right]\frac{d\xi_2}{dt} + \left[\frac{\omega_2}{\omega_0}\right]^2 \xi_2(1-\xi_0) + \left[\frac{\omega_2}{\omega_0}\right]^2 \xi_1\xi_2 = \left[\frac{\omega_2}{\omega_0}\right]^2 \bar{\xi}_2\xi_0 \qquad (15)$$

Next, we let

$$2\alpha_1\varepsilon = \frac{c_1\omega_1^2}{\omega_0}, 2\alpha_2\varepsilon = \frac{c_2\omega_2^2}{\omega_0}, Q = \frac{\omega_1}{\omega_0}, R = \frac{\omega_2}{\omega_0}, S = \left[\frac{\omega_2}{\omega_1}\right]^2 \qquad (16)$$

where,

$$\varepsilon = \lambda Q^2 = \lambda\left[\frac{\omega_1}{\omega_0}\right]^2, \qquad (17)$$

And

$$0 < \alpha_1 < 1,\ 0 < \alpha_2 < 1,\ 0 < Q < 1,\ 0 < R < 1 \ and\ 0 < \varepsilon < 1$$

Substituting (16) into (14) and (15) yield

$$\frac{d^2\xi_0}{dt^2} + \xi_0 = \lambda$$

(18) $\frac{d^2\xi_1}{dt^2} + 2\alpha_1\varepsilon\frac{d\xi_1}{dt} + Q^2\xi_1(1-\xi_0) - k_1Q^2\xi_1^2 + k_2Q^2\xi_2^2 = Q^2\bar{\xi}_1\xi_0$

(19)

$$\frac{d^2\xi_2}{dt^2} + 2\alpha_2\varepsilon\frac{d\xi_2}{dt} + R^2\xi_2(1-\xi_0) + R^2\xi_1\xi_2 = R^2\bar{\xi}_2\xi_0$$

(20) $\xi_i(0) = \xi_i'(0) = o; i = 1, 2.$

As in $[1-3]$, we neglect the pre-buckling inertia term, so that from (18) we get

$$\xi_0 = \lambda \qquad (21)$$

On simplification, using (21), equations (19) and (20)

yield $\frac{d^2\xi_1}{dt^2} + 2\alpha_1\varepsilon\frac{d\xi_1}{dt} + Q^2\xi_1 - \varepsilon\xi_1 - k_1Q^2\xi_1^2 + k_2Q^2\xi_2^2 = \varepsilon\,\bar{\xi}_1$

(22)

and

$$\frac{d^2\xi_2}{dt^2} + 2\alpha_2\varepsilon\frac{d\xi_2}{dt} + R^2\xi_2 - \varepsilon S\xi_2 + R^2\xi_1\xi_2 = \varepsilon S\,\bar{\xi}_2$$

(23)

$\xi_i(0) = \xi_i^{\prime}(0) = 0;\ i = 1, 2$

Where,

$$S = \left[\frac{R}{Q}\right]^2.$$

We assume a small time scale τ such that,

$$\tau = \varepsilon t \qquad (24a)$$

And

$$\xi_i^{\prime} = \xi_{i,t} + \varepsilon\xi_{i,\tau} \qquad (24b)$$

$$\xi_i^{\prime\prime} = \xi_{i,tt} + 2\varepsilon\xi_{i,t\tau} + \varepsilon^2\xi_{i,\tau\tau};\ i = 1,2 \qquad (24c)$$

We denote our perturbation parameter by ε so that

$$\xi_1(t) = \sum_{i=1}^{\infty}\varsigma^{(i)}(t,\tau)\varepsilon^i \qquad (25)$$

$$\xi_2(t) = \sum_{i=1}^{\infty}\eta^{(i)}(t,\tau)\varepsilon^i \qquad (26)$$

Substituting (25) and (26) into (22) and (23), using (24b) and (24c), equating terms of the orders of ε we get,

$$\varsigma_{,tt}^{(1)} + Q^2\varsigma^{(1)} = \bar{\xi}_1$$

(27) $\varsigma_{,tt}^{(2)} + Q^2\varsigma^{(2)} = -2\alpha_1\varsigma_{,t}^{(1)} + \varsigma^{(1)} + k_1Q^2\varsigma^{(1)2} - k_2Q^2\eta^{(1)2} - 2\varsigma_{,t\tau}^{(1)}$

(28)

and

$$\eta_{,tt}^{(1)} + R^2\eta^{(1)} = S\,\bar{\xi}_2$$

(29) $\eta_{,tt}^{(2)} + R^2\eta^{(2)} = -2\alpha_2\eta_{,t}^{(1)} + S\eta^{(1)} - 2\eta_{,t\tau}^{(1)} - R^2\varsigma^{(1)}\eta^{(1)}$

(30) $\varsigma^{(i)}(0,0)=\eta^{(i)}(0,0)=0, i=1,2$

(31) $\varsigma_{,t}^{(i)}(0,0)=\eta_{,t}^{(i)}(0,0)=0, i=1,2$

(32) $\varsigma_{,t}^{(i+1)}(0,0)+\varsigma_{,\tau}^{(i)}(0,0)=\eta_{,t}^{(i+1)}(0,0)+\eta_{,\tau}^{(i)}(0,0)=0, i=1,2$

(33)

The solution of (27) using (31) and (32) is

$$\varsigma^{(1)}(t,\tau)=a_1(\tau)\cos Qt+b_1(\tau)\sin Qt+\frac{\bar{\xi}_1}{Q^2}$$

(34a) $a_1(0)=-\frac{\bar{\xi}_1}{Q^2}; b_1(0)=0$

(34b)

Similarly, the solution of (28) is

$$\eta^{(1)}(t,\tau)=a_2(\tau)\cos Rt+b_2(\tau)\sin Rt+\frac{S\bar{\xi}_2}{R^2}$$

(35a) $a_2(0)=-\frac{S\bar{\xi}_2}{R^2}; b_2(0)=0$

(35b)

Substituting using (34a) and (35a) into (28), we

have $\varsigma_{,tt}^{(2)}+Q^2\varsigma^{(2)}=-2\alpha_1[-Qa_1\sin Q\mathrm{t}+Q\mathrm{b}_1\cos Q\mathrm{t}]+\left[a_1\cos Q\mathrm{t}+\mathrm{b}_1\sin Q\mathrm{t}+\frac{\bar{\xi}_1}{Q^2}\right]$

$$-\mathrm{k}_2Q^2\left[\frac{1}{2}[a_2^2+b_2^2]+a_2b_2\sin 2\mathrm{Rt}+\frac{1}{2}[a_2^2-b_2^2]\cos 2\mathrm{Rt}\right]$$

$$-\mathrm{k}_2Q^2\left[+\frac{2a_2\mathrm{S}\bar{\xi}_2}{\mathrm{R}^2}\cos\mathrm{Rt}+\frac{2b_2\mathrm{S}\bar{\xi}_2}{\mathrm{R}^2}\sin\mathrm{Rt}\right]$$

$$+\mathrm{k}_1 Q^2\left[\frac{1}{2}[a_1^2+b_1^2]+\mathrm{a}_1\mathrm{b}_1\sin 2Q\mathrm{t}+\frac{1}{2}[a_1^2-b_1^2]\cos 2Q\mathrm{t}\right]$$

$$+\mathrm{k}_1 Q^2\left[+\frac{2\,\mathrm{a}_1\,\bar{\xi}_1}{Q^2}\cos Q\,\mathrm{t}+\frac{2\,\mathrm{b}_1\,\bar{\xi}_1}{Q^2}\sin Q\,\mathrm{t}\right]+2Q\left[\mathrm{a}_1'\sin Q\,\mathrm{t}-\mathrm{b}_1'\cos Q\,\mathrm{t}\right] \qquad (36)$$

Now, to ensure a uniformly valid asymptotic solution in t, we equate to zero, in (36), the coefficients of cos Q t and Sin Q t to get

$$b_1' + \alpha_1 b_1 = a_1\varphi \qquad (37a)$$

And

$$a_1' + \alpha_1 a_1 = -b_1\varphi \qquad (37b)$$

where,

$$(\ )' = \frac{d(\ )}{d\tau},$$

$$\varphi = \frac{1}{2Q}\left[1+2k_1\,\bar{\xi}_1\right]$$

Simplification of (37a, b) yield

$$b_1'' + \alpha_1 b_1' = -\varphi\left[b_1\left[\varphi+\frac{\alpha_1^2}{\varphi}\right]+\frac{\alpha_1 b_1'}{\varphi}\right]$$

$$b_1'' + 2\alpha_1 b_1' + \varphi b_1\left[\varphi+\frac{\alpha_1^2}{\varphi}\right]=0$$

$$b_1(0)=0;\, b_1'(0)=-\frac{\varphi\bar{\xi}_1}{Q^2} \qquad (37c)$$

And

$$a_1'' + \alpha_1 a_1' = -\,\varphi\left[a_1\left[\varphi+\frac{\alpha_1^2}{\varphi}\right]+\frac{\alpha_1 a_1'}{\varphi}\right]$$

$$a_1'' + 2\alpha_1 a_1' + \varphi a_1\left[\varphi+\frac{\alpha_1^2}{\varphi}\right]=0$$

$$a_1(0)=-\frac{\bar{\xi}_1}{Q^2};\, a_1'(0)=\frac{\alpha_1\,\bar{\xi}_1}{Q^2} \qquad (37d)$$

The remaining part of the equation in the substitution into (28) as obtained from (36) is

$$\varsigma_{,tt}^{(2)} + Q^2\varsigma^{(2)} = q_1 + k_1 Q^2\left[p_0(\tau)\sin 2Qt + p_1(\tau)\cos 2Qt\right]$$

$$-k_2Q^2\left[p_2(\tau)\sin 2Rt + p_3(\tau)\cos 2Rt + p_4(\tau)\cos Rt + p_5(\tau)\sin Rt\right] \quad (38a)$$

$$\varsigma^{(2)}(0,0) = 0; \varsigma_{,t}^{(2)}(0,0) + \varsigma_{,\tau}^{(2)}(0,0) = 0 \quad (38b)$$

Where

$$q_1 = \frac{\bar{\xi}_1}{Q^2} + k_1Q^2 r_0(\tau) - k_2Q^2 r_1(\tau); p_0(\tau) = a_1b_1; p_1(\tau) = \frac{1}{2}\left[a_1^2 - b_1^2\right]$$

(38c) $p_2(\tau) = a_2b_2; p_3(\tau) = \frac{1}{2}\left[a_2^2 - b_2^2\right]; p_4(\tau) = \frac{2a_2S\bar{\xi}_2}{R^2}; p_5(\tau) = \frac{2b_2S\bar{\xi}_2}{R^2}$

(38d) $r_0(\tau) = \frac{1}{2}\left[a_2^2 + b_2^2\right]; r_1(\tau) = \frac{1}{2}\left[a_1^2 + b_1^2\right]$

(38e) $p_0(0) = 0; p_1(0) = \frac{\bar{\xi}_1^2}{2Q^4}; p_2(0) = 0; p_3(0) = \frac{S^2\bar{\xi}_2^2}{2R^4}$

(38f) $p_4(0) = \frac{2S^2\bar{\xi}_2^2}{R^4}; p_5(0) = 0; r_0(0) = \frac{\bar{\xi}_1^2}{2Q^4}; r_1(0) = \frac{S^2\bar{\xi}_2^2}{2R^4};$

(38g)

The solution of (38a), using (36b)

is $\varsigma^{(2)}(t,\tau) = a_3(\tau)\cos Qt + b_3(\tau)\sin Qt + \frac{q_1}{Q^2} - \frac{k_1}{3}\left[p_6(\tau)\sin 2Qt + p_7(\tau)\cos 2Qt\right]$

$$-k_2Q^2\left[p_8(\tau)\sin 2Rt + p_9(\tau)\cos 2Rt + p_{10}(\tau)\cos Rt + p_{11}(\tau)\sin Rt\right] \quad (39a)$$

$$a_3(0) = \bar{\xi}_1 l_0 + k_1\bar{\xi}_1^2 l_1 + k_2\bar{\xi}_2^2\left[\frac{S}{R^2}\right]^2 l_2; b_3(0) = -\frac{\alpha_1\bar{\xi}_1}{Q^3} \quad (39b)$$

Where

$$l_0 = -\frac{1}{Q^4}; l_1 = -\frac{1}{2Q^2} + \frac{1}{6Q^4}; l_2 = \frac{1}{2} + \frac{1}{2\left[Q^2 - 4R^2\right]} - \frac{2}{R^2\left[Q^2 - 4R^2\right]}$$

(39c) $p_6(\tau) = a_1b_1; p_7(\tau) = a_2b_2; p_8(\tau) = \frac{p_2(\tau)}{Q^2 - 4R^2}; p_9(\tau) = \frac{p_3(\tau)}{Q^2 - 4R^2}$

(39d) $p_{10}(\tau) = \dfrac{p_4(\tau)}{Q^2 - R^2}; p_{11}(\tau) = \dfrac{p_5(\tau)}{Q^2 - R^2}; p_{11}(0) = 0$

(39e)

$$p_6(0) = 0; p_7(0) = \frac{\bar{\xi}_1^{\,2}}{2Q^4}; p_8(0) = 0; p_9(0) = \frac{S^2 \xi_2^2}{2R^4\left[Q^2 - 4R^2\right]}; p_{10}(0) = \frac{2S^2 \bar{\xi}_2^{\,2}}{R^4\left[Q^2 - R^2\right]} \quad \text{(39f)}$$

Substituting using (34a) and (35a) into (30) we get

$$\eta^{(2)}_{,tt} + R^2 \eta^{(2)} = -2\alpha_2\left[-Ra_2 \sin Rt + Rb_2 \cos Rt\right] - 2R\left[-a_2' \sin Rt + b_2' \cos Rt\right] + S\left[a_2 \cos Rt + b_2 \sin Rt + \frac{S\xi_2}{R^2}\right]$$

$$-\frac{R^2}{2}\left[\begin{array}{l} \dfrac{2S\bar{\xi}_1\bar{\xi}_2}{[QR]^2} + \dfrac{2a_2\bar{\xi}_1}{Q^2}\cos Rt + \dfrac{2b_2\bar{\xi}_1}{Q^2}\sin Rt + \dfrac{2a_1 S\bar{\xi}_2}{R^2}\cos Qt + \dfrac{2b_1 S\bar{\xi}_2}{R^2}\sin Qt \\ +[a_1a_2 - b_1b_2]\cos[Q-R]t + [a_1b_2 + b_1a_2]\sin[Q+R]t \\ +[a_1a_2 + b_1b_2]\cos[Q-R]t + [b_1a_2 - a_1b_2]\sin[Q-R]t \end{array}\right] \quad (40)$$

Now, to ensure a uniformly valid asymptotic solution in t, we equate the coefficients of cosRt and SinRt to zero so that

$$b_2' + \alpha_2 b_2 = a_2\Phi \quad \text{(41a)}$$

and

$$a_2' + \alpha_2 a_2 = -b_2\Phi \quad \text{(41b)}$$

where

$$\Phi = \frac{1}{2R}\left[S - \frac{R^2\bar{\xi}_1}{Q^2}\right].$$

Simplification of (41) yields

$$b_2'' + \alpha_2 b_2' = -\Phi\left[\Phi b_2 + \alpha_2 a_2\right]$$

$$b_2'' + \alpha_2 b_2' = -\Phi\left[\Phi b_2 + \frac{\alpha_2}{\Phi}\left[b_2' + \alpha_2 b_2\right]\right]$$

$$b_2^{//} + 2\alpha_2 b_2^{/} + b_2\left[\Phi^2 + \alpha_2^2\right] = 0$$

$$b_2(0) = 0; b_2^{/}(0) = -\frac{\Phi S\,\bar{\xi}_2}{R^2} \tag{41c}$$

$$b_2(0) = 0; b_2^{/}(0) = -\frac{\Phi S\,\bar{\xi}_2}{R^2}$$

$$a_2^{//} + \alpha_2 a_2^{/} = -\,\Phi\left[\Phi a_2 - \alpha_2 b_2\right]$$

$$a_2^{//} + \alpha_2 a_2^{/} = -\Phi\left[\Phi a_2 + \frac{\alpha_2}{\Phi}\left[a_2^{/} + \alpha_2 a_2\right]\right]$$

$$a_2^{//} + 2\alpha_2 a_2^{/} + a_2\left[\Phi^2 + \alpha_2^2\right] = 0$$

$$a_2(0) = -\frac{S\,\bar{\xi}_2}{R^2}; a_2^{/}(0) = -\frac{\alpha_2 S\,\bar{\xi}_2}{R^2} \tag{41d}$$

The remaining part of the equation in the substitution into (30) as obtained from (40) is

$$\eta_{,tt}^{(2)} + R^2\eta^{(2)} = q_2 - \frac{R^2}{2}\left[\begin{matrix} p_{12}(\tau)\cos Qt + p_{13}(\tau)\sin Qt + p_{14}(\tau)\cos[Q+R]t \\ + p_{15}(\tau)\sin[Q+R]t + p_{16}(\tau)\cos[Q-R]t + p_{17}(\tau)\sin[Q-R] \end{matrix}\right]$$

$$\eta^{(2)}(0,0) = 0; \eta_{,t}^{(2)}(0,0) + \eta_{,\tau}^{(1)}(0,0) = 0 \tag{42b}$$

where,

$$q_2 = \frac{S^2\,\bar{\xi}_2}{R^2} - \frac{S\,\bar{\xi}_1\,\bar{\xi}_2}{Q^2}; p_{12}(\tau) = \frac{2a_1 S\,\bar{\xi}_2}{R^2}; p_{13}(\tau) = \frac{2b_1 S\,\bar{\xi}_2}{R^2}; p_{14}(\tau) = a_1 a_2 - b_1 b_2 \tag{42c}$$

$$p_{15}(\tau) = b_1 b_2 + b_1 a_2; p_{16}(\tau) = a_1 a_2 + b_1 b_2; p_{17}(\tau) = b_1 a_2 - a_1 b_1 \tag{42d}$$

$$p_{12}(0) = \frac{2S\,\bar{\xi}_1\,\bar{\xi}_2}{Q^2 R^2}; p_{13}(0) = 0; p_{14}(0) = \frac{S\,\bar{\xi}_1\,\bar{\xi}_2}{Q^2 R^2}; p_{15}(0) = 0; \tag{42e}$$

$$p_{16}(0) = \frac{S\,\bar{\xi}_1\,\bar{\xi}_2}{Q^2 R^2}; p_{17}(0) = 0 \tag{42f}$$

The solution of (42a) using (42b) is

$$\eta^{(2)}(t,\tau) = a_4(\tau)\cos Rt + b_4(\tau)\sin Rt + \frac{q_2}{R^2} - \frac{1}{2}\begin{bmatrix} p_{18}(\tau)\cos Qt + p_{19}(\tau)\sin Qt + \\ p_{20}(\tau)\cos[Q+R]t + p_{21}(\tau)\sin[Q+R]t + \\ p_{22}(\tau)\cos[Q-R]t + p_{23}(\tau)\sin[Q-R]t \end{bmatrix} \quad \text{(43a}$$

$$a_4(0) = S^2\,\bar{\xi}_2\,l_3 + R^2 S\,\bar{\xi}_1\,\bar{\xi}_2\,l_4 ; b_4(0) = -\frac{\alpha_2 S\,\bar{\xi}_2}{R^3} \quad \text{(43b)}$$

where

$$l_4 = \left[\frac{1}{\left[R^2 Q\right]^2} + \frac{1}{2}\left[\frac{-2}{[RQ]^2\left[R^2-Q^2\right]} - \frac{1}{Q[RQ]^2[2R+Q]} + \frac{1}{Q[RQ]^2[2R-Q]}\right]\right] \quad \text{(43c)}$$

$$l_3 = -\frac{1}{R^4} ; p_{18}(\tau) = \frac{p_{12}(\tau)}{R^2-Q^2} ; p_{19}(\tau) = \frac{p_{13}(\tau)}{R^2-Q^2} ; p_{20}(\tau) = \frac{p_{14}(\tau)}{Q[2R+Q]} \quad \text{(43d)}$$

$$p_{21}(\tau) = \frac{p_{15}(\tau)}{Q[2R+Q]} ; p_{22}(\tau) = \frac{p_{16}(\tau)}{Q[2R-Q]} ; p_{23}(\tau) = \frac{p_{17}(\tau)}{Q[2R-Q]} \quad \text{(43e)}$$

$$p_{18}(0) = \frac{2S\,\bar{\xi}_1\,\bar{\xi}_2}{Q^2R^2\left[R^2-Q^2\right]} ; p_{19}(0) = 0 ; p_{20}(0) = \frac{S\,\bar{\xi}_1\,\bar{\xi}_2}{Q^3R^2[2R+Q]} \quad \text{(43f)}$$

$$p_{21}(0) = 0 ; p_{22}(0) = \frac{S\,\bar{\xi}_1\,\bar{\xi}_2}{Q^2R^2[2R-Q]} ; p_{23}(0) = 0 \quad \text{(43g)}$$

Next, using (34), (35), (39) and (43) we deduce the displacements as

$\xi_1(t) = \varsigma^{(1)}(t,\tau)\varepsilon + \varsigma^{(2)}(t,\tau)\varepsilon^2 + \ldots$

(44a) $\xi_2(t) = \eta^{(1)}(t,\tau)\varepsilon + \eta^{(2)}(t,\tau)\varepsilon^2 + \ldots$

(44b)

We seek the maximum displacement for both $\xi_1(t)$ and $\xi_2(t)$. To achieve this, we shall first determine the critical values of t and τ for each of $\xi_1(t)$ and $\xi_2(t)$ at their maximum values. The conditions for the maximum displacement of $\xi_1(t)$ and $\xi_2(t)$ is obtain from (24b)

$$\xi_{1,t} + \varepsilon\xi_{1,\tau} , \quad \text{(45a)}$$

$$\xi_{2,t} + \varepsilon \xi_{2,\tau}, \tag{45b}$$

We know from (44a, b) that

$$\xi_1(t) = \varsigma^{(1)}(t,\tau)\varepsilon + \varsigma^{(2)}(t,\tau)\varepsilon^2 + \ldots \tag{46a}$$

$$\xi_2(t) = \eta^{(1)}(t,\tau)\varepsilon + \eta^{(2)}(t,\tau)\varepsilon^2 + \ldots \tag{46b}$$

On applying (45a, b) to (46a, b), we get

$$\varsigma_{,t} + \varepsilon \varsigma_{,\tau} = \left[\varsigma_{,t}^{(1)}(t_a,\tau_a)\varepsilon + \varsigma_{,t}^{(2)}(t_a,\tau_a)\varepsilon^2 + \ldots\right]$$

$$+\varepsilon\left[\varsigma_{,\tau}^{(1)}(t_a,\tau_a)\varepsilon + \varsigma_{,\tau}^{(2)}(t_a,\tau_a)\varepsilon^2 + \ldots\right] = 0 \tag{47a}$$

And

$$\eta_{,t} + \varepsilon \eta_{,\tau} = \left[\eta_{,t}^{(1)}(T_c,\tau_c)\varepsilon + \eta_{,t}^{(2)}(T_c,\tau_c)\varepsilon^2 + \ldots\right]$$

$$+\varepsilon\left[\eta_{,\tau}^{(1)}(T_c,\tau_c)\varepsilon + \eta_{,\tau}^{(2)}(T_c,\tau_c)\varepsilon^2 + \ldots\right] = 0 \tag{47b}$$

where, (t_a,τ_a) and (T_c,τ_c) are the values of t and τ at the maximum displacement of $\varsigma(t,\tau)$ and $\eta(t,\tau)$ respectively.

We now expand (47a, b) in a Taylor series about $t_a = t_0, \tau_a = 0$ and $T_c = T_0, \tau_c = 0$,and thereafter equate to zero the terms of the same orders of ε.

$$\varsigma_{,t}^{(1)}(t_0,0) = 0 \tag{48a}$$

$$t_1\varsigma_{,tt}^{(1)}(t_0,0) + t_0\varsigma_{,t\tau}^{(1)}(t_0,0) + \varsigma_{,t}^{(2)}(t_0,0) + \varsigma_{,\tau}^{(1)}(t_0,0) = 0 \tag{48b}$$

and

$$\eta_{,t}^{(1)}(T_0,0) = 0 \tag{49a}$$

$$T_1\eta_{,tt}^{(1)}(T_0,0) + T_0\eta_{,t\tau}^{(1)}(T_0,0) + \eta_{,t}^{(2)}(T_0,0) + \eta_{,\tau}^{(1)}(T_0,0) = 0 \tag{49b}$$

Substituting for $\varsigma_{,t}^{(1)}$ from (34a) in (48a) and simplifying we get

$$\sin Qt_0 = 0 \tag{50a}$$

A further simplification of (50a) gives

$$t_0 = \frac{\pi}{Q} \tag{50b}$$

A similar solution for (49a) is

$$T_0 = \frac{\pi}{R} \tag{50c}$$

Next, we deduce from (48b) that

$$t_1 = -\frac{1}{\varsigma_{,tt}^{(1)}(t_0,0)}\left[t_0\varsigma_{,t\tau}^{(1)}(t_0,0) + \varsigma_{,t}^{(2)}(t_0,0) + \varsigma_{,\tau}^{(1)}(t_0,0)\right] \tag{51a}$$

Simplification of the following terms are however necessary in this analysis,

$$\varsigma_{,t}^{(2)}(t_0,0) = \alpha_1\bar{\xi}_1 l_5 + k_1\bar{\xi}_1^2 l_6 - k_2 S^2\bar{\xi}_2^2 l_7; \varsigma_{,t}^{(1)}(t_0,0) = \bar{\xi}_1 l_8 \tag{51b}$$

$$\varsigma_{,\tau}^{(1)}(t_0,0) = \bar{\xi}_1 l_9; \varsigma_{,tt}^{(1)}(t_0,0) = \bar{\xi}_1; \varsigma^{(1)}(t_0,0) = 2\bar{\xi}_1 l_{10} \tag{51c}$$

$$\varsigma^{(2)}(t_0,0) = 2\bar{\xi}_1 l_{11} + k_1\bar{\xi}_1^2 l_{12} + k_2\bar{\xi}_2^2\left[\frac{S}{R^2}\right]^2 l_{13}; \varsigma_{,t}^{(1)}(t_0,0) = 0; \tag{51d}$$

where

$$l_5 = \frac{1}{Q^2}; l_6 = \frac{Sin2Qt_0}{3Q^2}; l_7 = \frac{Q^2 Sin2Rt_0}{R^3\left[Q^2 - 4R^2\right]} \tag{51e}$$

$$l_8 = \frac{\varphi}{Q}; l_9 = -\frac{\alpha_1}{Q^2}; l_{10} = \frac{1}{Q^2}; l_{11} = \frac{1}{Q^4}; l_{12} = \frac{1}{Q^2} - \frac{1}{3Q^4} \tag{51f}$$

$$l_{13} = \left[-1 - \frac{1}{2\left[Q^2 - 4R^2\right]} + \frac{1}{R^2\left[Q^2 - 4R^2\right]} - Q^2\left[\frac{1}{Q^2 - 4R^2} + \frac{2}{Q^2 - R^2}\right]\right] \tag{51g}$$

On substituting (51, b-d) on (51a), we have

$$t_1 = \alpha_1 l_5 + k_1\bar{\xi}_1 l_6 - k_2 S\bar{\xi}_2 l_7 + t_0 l_8 + l_9 \tag{52}$$

Similarly, deducing from (49b) yields

$$T_1 = -\frac{1}{\eta_{,tt}^{(1)}(T_0,0)}\left[T_0\eta_{,t\tau}^{(1)}(T_0,0) + \eta_{,t}^{(2)}(T_0,0) + \eta_{,\tau}^{(1)}(t_0,0)\right] \tag{53a}$$

We however note the following simplifications

$$\eta_{,t}^{(2)}(T_0,0) = \alpha_2 S\bar{\xi}_2 l_{14} + S^2\bar{\xi}_2 l_{15} + S\bar{\xi}_1\bar{\xi}_2 l_{16}; \eta_{,t\tau}^{(1)}(T_0,0) = S\bar{\xi}_2 l_{17} \tag{53b}$$

$$\eta_{,\tau}^{(1)}(T_0,0) = S^2\bar{\xi}_2 l_{18}; \eta_{,tt}^{(1)}(T_0,0) = -S\bar{\xi}_2; \eta^{(1)}(T_0,0) = 2S\bar{\xi}_2 l_{20} \tag{53c}$$

$$\eta^{(2)}(T_0,0) = S^2\,\bar{\xi}_2\, l_{21} + R^2 S\,\bar{\xi}_1\,\bar{\xi}_2\, l_{19}; \eta^{(1)}_{,t}(T_0,0) = 0; \tag{53d}$$

where

$$l_{14} = -\frac{\cos RT_0}{R^2}; l_{15} = -Rl_3 \sin RT_0 \tag{53e}$$

$$l_{16} = -R^3 S l_4 \sin RT_0 - \frac{R^2}{2}\left[\begin{array}{l}\dfrac{2\sin QT_0}{QR^2\left[R^2-Q^2\right]} - \dfrac{\cos QT_0}{[RQ]^2[2R+Q]} \\ -\dfrac{[Q+R]\sin[Q+R]T_0}{Q[RQ]^2[2R+Q]} - \dfrac{[Q-R]\sin[Q-R]T_0}{Q[RQ]^2[2R-Q]}\end{array}\right] \tag{53f}$$

$$l_{17} = \frac{\Phi}{R}; l_{18} = -\frac{\alpha_2}{R^2}; l_{20} = \frac{1}{R^2}; l_{21} = \frac{1}{R^4}; l_{21} = \frac{1}{R^4} \tag{53g}$$

$$l_{19} = \left[-l_4 + \frac{1}{2}\left[\frac{2\cos QT_0}{Q^2R^2\left[R^2-Q^2\right]} + \frac{\cos[Q+R]T_0}{Q[RQ]^2[2R+Q]} + \frac{\cos[Q-R]T_0}{Q[RQ]^2[2R-Q]}\right]\right] \tag{53h}$$

On substituting (53, b-d) on (53a), we have

$$T_1 = \alpha_2 l_{14} + \bar{\xi}_1\, l_{16} + T_0 l_{17} + l_{18} \tag{54}$$

We, now, determine the maximum values of $\varsigma(t)$ and $\eta(t)$ say ς_a and η_c respectively by evaluating (46 a, b) at the critical values namely $t = t_a, \tau = \tau_a$ and $T = T_c, \tau = \tau_c$.

$$\varsigma_a = \varsigma^{(1)}(t_a,\tau_a)\varepsilon + \varsigma^{(2)}(t_a,\tau_a)\varepsilon^2 + \ldots \tag{55a}$$

$$\eta_c = \eta^{(1)}(T_c,\tau_c)\varepsilon + \eta^{(2)}(T_c,\tau_c)\varepsilon^2 + \ldots \qquad \text{(55b}$$

Expanding (55 a) in Taylor series using,

$$t_a = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \ldots; \tau_a = \varepsilon t_a = \varepsilon\left[t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \ldots\right] \tag{56a}$$

we have

$$\varsigma_a = \varepsilon\left[\varsigma^{(1)}(t_0,0) + \varsigma^{(1)}_{,t}(t_0,0)\left[\varepsilon t_1 + \varepsilon^2 t_2 + \ldots\right] + \varsigma^{(1)}_{,\tau}(t_0,0)\varepsilon\left[t_0 + t_1\varepsilon_1 + \ldots\right]\right]$$

$$+\varsigma^{(2)}(t_0,0)\varepsilon^2 + \ldots \tag{56b}$$

Regrouping the terms in orders of ε yields

$$\varsigma_a = \varepsilon\varsigma^{(1)}(t_0,0) + \varepsilon^2\left[t_1\varsigma^{(1)}_{,t}(t_0,0) + t_0\varsigma^{(1)}_{,\tau}(t_0,0) + \varsigma^{(2)}(t_0,0)\right] + \ldots \tag{56c}$$

On substituting the terms in (56c) from (51, b-d), we have

$$\varsigma_a = 2\,\bar{\xi}_1\, l_{10}\varepsilon + \left[t_0\,\bar{\xi}_1\, l_9 + 2\,\bar{\xi}_1\, l_{11} + k_1\,\bar{\xi}_1^2\, l_{12} + k_2\,\bar{\xi}_2^2\left[\frac{S}{R^2}\right]^2 l_{13}\right]\varepsilon^2 + \ldots \tag{57}$$

Similarly, expanding (55 b) in Taylor series using,

$$T_c = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + ...; \tau_c = \varepsilon T_c = \varepsilon\left[T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + ...\right] \quad (58a)$$

we have

$$\eta_c = \varepsilon\left[\eta^{(1)}(T_0,0) + \eta^{(1)}_{,t}(T_0,0)\left[\varepsilon T_1 + \varepsilon^2 T_2 + ...\right] + \eta^{(1)}_{,\tau}(T_0,0)\varepsilon\left[T_0 + \varepsilon_1 T_1 + ...\right]\right]$$

$$+\eta^{(2)}(T_0,0)\varepsilon^2 + ... \quad (58b)$$

Regrouping the terms in orders of ε yields

$$\eta_c = \varepsilon\eta^{(1)}(T_0,0) + \varepsilon^2\left[T_1\eta^{(1)}_{,t}(T_0,0) + T_0\eta^{(1)}_{,\tau}(T_0,0) + \eta^{(2)}(T_0,0)\right] + ... \quad (58c)$$

On substituting the terms in (58c) from (53, b-d), we have

$$\eta_c = 2S\bar{\xi}_2 l_{20}\varepsilon + \left[T_0 S^2 \bar{\xi}_2 l_{18} + S^2 \bar{\xi}_2 l_{21} + R^2 S \bar{\xi}_1 \bar{\xi}_2 l_{19}\right]\varepsilon^2 + ... \quad (59)$$

The net maximum displacement ξ_m is

$$\xi_m = \varsigma_a + \eta_c = \varsigma(t_a, \tau_a) + \eta(T_c, \tau_c) \quad (60)$$

Substituting for terms in (60) from (57) and (59) we get

$$\xi_m = C_1\varepsilon + C_2\varepsilon^2 + ... \quad (61a)$$

where

$$C_1 = l_{22}; C_2 = \bar{\xi}_1 l_{23} + S^2 \bar{\xi}_2 l_{24} + k_1 \bar{\xi}_1^2 l_{12} + k_2 \bar{\xi}_2^2 \left[\frac{S}{R^2}\right]^2 l_{13} + R^2 \bar{\xi}_1 \bar{\xi}_2 S l_{19} \quad (61b)$$

$$l_{22} = 2\bar{\xi}_1 l_{10} + 2S\bar{\xi}_2 l_{20}; l_{23} = 2l_{11} + t_0 l_9; l_{24} = l_{21} + T_0 l_{18} \quad (61c)$$

As noted by $[1-3]$ and $[21]$,the condition for dynamic buckling is

$$\frac{d\lambda}{d\xi_m} = 0 \quad (62)$$

As in $[23-24]$, applying the method of reversal of series of (61a), we get

$$\varepsilon = d_1\xi_m + d_2\xi_m^2 + ... \quad (63)$$

Substituting for ξ_m from (61a) in (63) and equating powers of orders of ε, we get $d_1 = \frac{1}{C_1}, d_2 = -\frac{C_2}{C_1^3}$

(64)

The maximization in (62) is better done from (63), thus implementing (62) using (63) we have

$$\xi_m(\lambda_D) = \frac{C_1^2}{2C_2} \quad (65)$$

where, $\xi_m(\lambda_D)$ is the value of the net displacement at buckling. In determining the dynamic buckling load, we evaluate (63) at

$\lambda = \lambda_D$

to yield

$$\varepsilon = \xi_m(\lambda_D)\left[d_1 + d_2\xi_m\right]_{(\lambda=\lambda_D)} \quad (66)$$

On substituting for terms d_1 and d_2 from (64) and $\xi_m(\lambda_D)$ from (65) in (66) and simplify to get

$$\varepsilon\lambda_D = \frac{C_1}{4C_2} \quad (67)$$

The expansion of (67) gives [using (61b, c)]

$$\lambda_D = \frac{1}{4}\left[\frac{\omega_0}{\omega_1}\right]^2 \left[\left[2\bar{\xi}_1 l_{10} + 2S\,\bar{\xi}_2 l_{20}\right]\right] \left[\left[\begin{array}{l} \bar{\xi}_1 l_{23} + S^2\,\bar{\xi}_2 l_{24} + k_1 \bar{\xi}_1^2 l_{12} + k_2 \bar{\xi}_2^2 \left[\frac{S}{R^2}\right]^2 l_{13} \\ + R^2\,\bar{\xi}_1\,\bar{\xi}_2 S\, l_{19} \end{array}\right]^{-1}\right] \quad (68)$$

Here, (68) gives the formula for evaluating the dynamic buckling load λ_D , and is valid for $R \neq (1, 2, Q, 2Q, 1-Q, 1+Q)$ and $Q \neq (R, 2R, 1-R, 1+R, 0, 2R-1)$

4. ANALYSIS OF RESULT

The above results indicate that dynamic buckling load increases if the structure is less imperfect. The results also show that dynamic buckling load increases with increased damping. In addition, the results confirm that the only condition in which the effect of the coupling between the buckling modes is felt is if none of the imperfection parameters in the shape of the mode coupling is neglected. Once an imperfection is neglected the coupling effect of the mode that is in the shape of the neglected imperfection, with any other mode is neglected. For a graphical view of this phenomenon, we use the following values. k_1=0.2, k_2=0.3, $\bar{\xi}_1$ =0.01, $\bar{\xi}_2$ =0.03, α_1=0.01 and α_2=0.03. By varying $\bar{\xi}_1$ and α_1 while keeping $\bar{\xi}_2$ constant at 0.03 and α_2 = 0, the corresponding values of λ_D were computed from (68). The plots of dynamic buckling load against the imperfection parameter and light viscous damping of the discretized spherical cap are shown in figures 1 and 2 below. We represent

$\bar{\xi}_1$ by Zih1bar, $\bar{\xi}_2$ by Zih2bar, α_1 by Alpha1, α_2 by Alpha2 and λ_D by LambdaD on the figures.

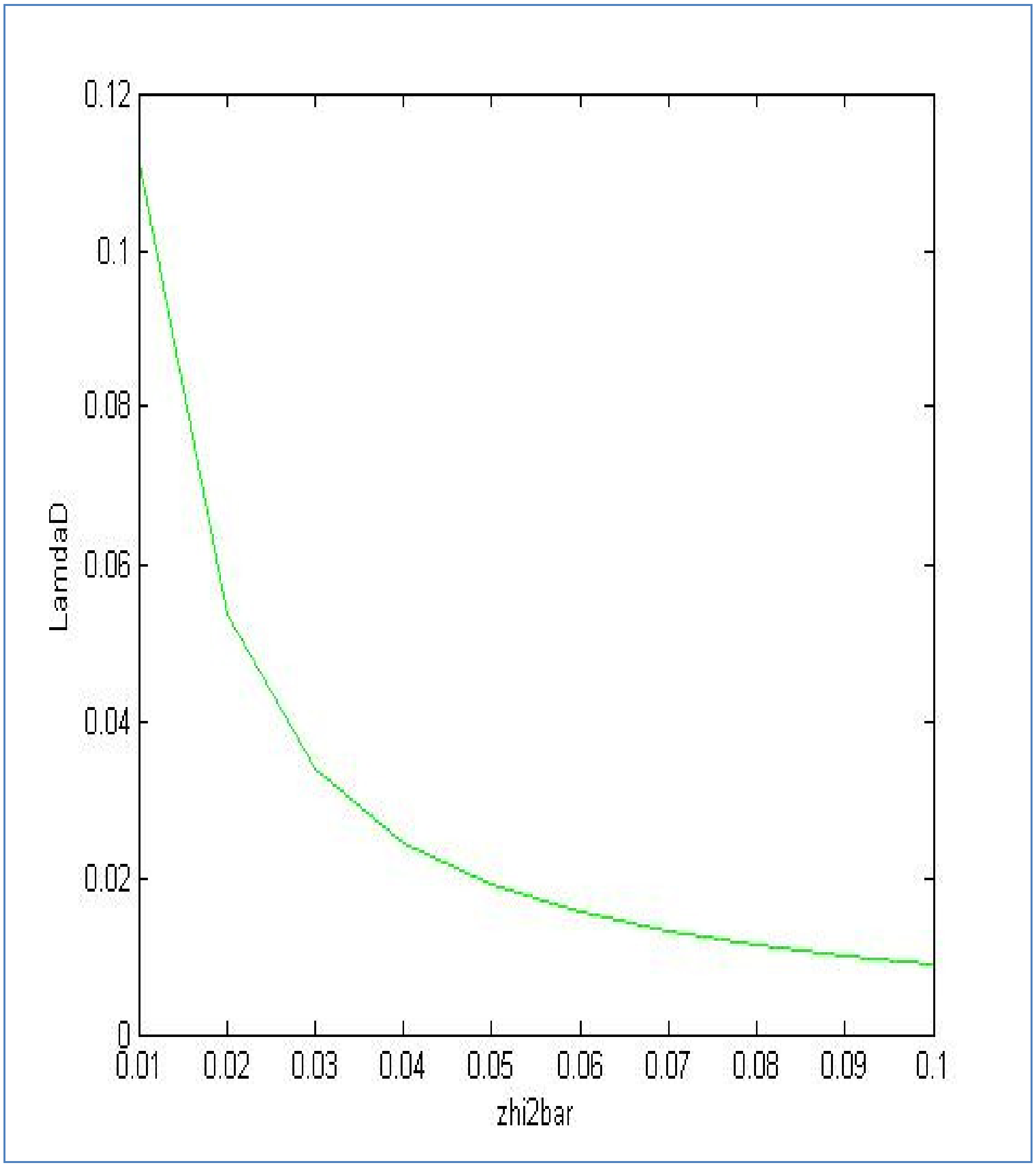

Figure 1: Dynamic buckling load of a spherical cap against the imperfection parameter $\bar{\xi}_2$ ($\bar{\xi}_1 = 0.02$)

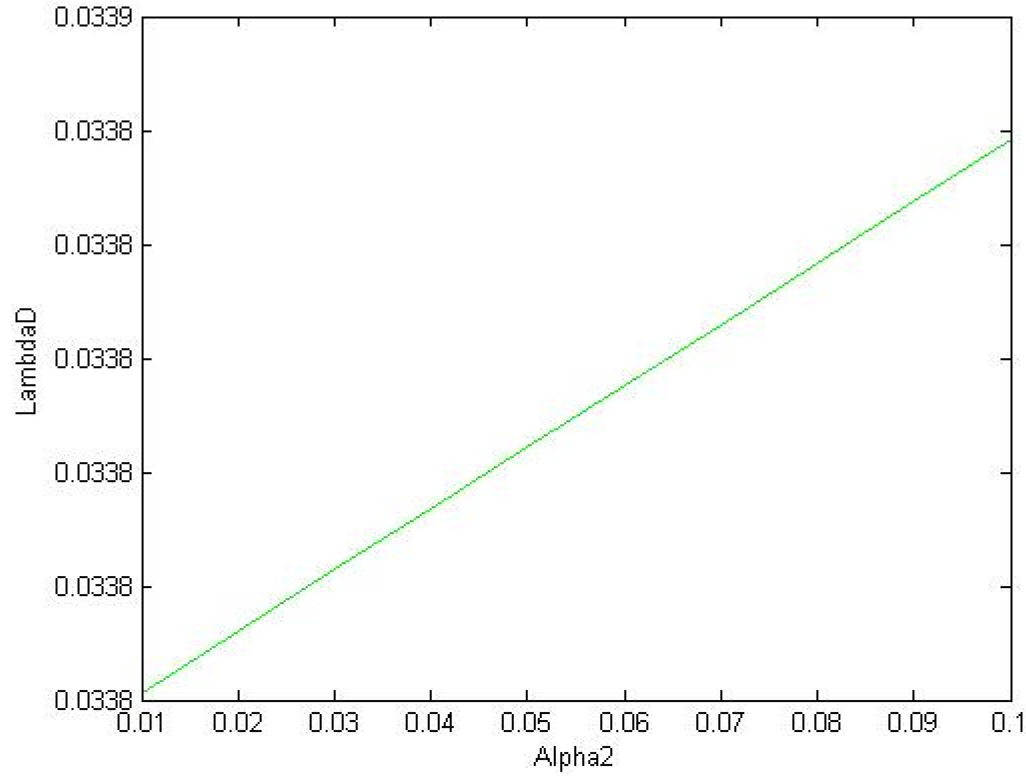

Figure 2: Dynamic buckling load of a spherical cap against the light viscous damping $\alpha_2(\alpha_1 = 0)$

We note that the results display all the imperfection parameters stated in problems (3)-(5).This is unlike Danielson's problem in which the axisymmetric imperfection was neglected for easy solution. In fact, the method is such that we can adequately account for all modal imperfections allowed in the formulation .The contributions of the quadratic terms $k_1\xi_1^2, k_2\xi_2^2$ and the coupling term $\xi_1\xi_2$ are respectively given in the denominator of (68) by $k_1\bar{\xi}_1^2 l_{11}, k_2\bar{\xi}_2^2\left[\frac{S}{R^2}\right]l_{13}$ and $R^2\bar{\xi}_1\bar{\xi}_2 Sl_{19}$.Thus if we assume that the axysymmetric imperfections are zero then $\bar{\xi}_1 = 0,$ and the dynamic buckling load λ_D responsible for the buckling in this case is obtained from (68) as

$$\lambda_D = \frac{1}{4}\left[\frac{\omega_0}{\omega_1}\right]^2 \left|\left[2S\bar{\xi}_2 l_{20}\right]\right| \left|\left[S^2\bar{\xi}_2 l_{24} + k_2\bar{\xi}_2^2\left[\frac{S}{R^2}\right]^2 l_{13}\right]^{-1}\right| \quad (69)$$

We note from (69), that, the effect of the coupling terms $\xi_1\xi_2$, $\xi_1\xi_0$ and $k_1\xi_1^2$ are zeros. The effect of the quadratic term $k_2\xi_2^2$ is non-zero and it is this term that dominates the buckling process. Neglecting $\bar{\xi}_1$ is sufficient to completely nullify the effect of ξ_1^2 where the converse is not necessarily the case. However, if the non-axymmetric imperfections are neglected then $\bar{\xi}_2 = 0,$ and the dynamic buckling load λ_D following (68) become

$$\lambda_D = \frac{1}{4}\left[\frac{\omega_0}{\omega_1}\right]^2 \left|\left[2\bar{\xi}_1 l_{10}\right]\right| \left|\left[\bar{\xi}_1 l_{23} + k_1 \bar{\xi}_1^2 l_{12}\right]^{-1}\right| \tag{70}$$

We deduce from (70), that, the effect of the coupling terms $\xi_1\xi_2$, $\xi_2\xi_0$ and $k_2\xi_2^2$ are again zeros. The effect of the quadratic term $k_1\xi_1^2$ is non-zero and this singular term is the only non-linear term that influences the buckling process. Neglecting $\bar{\xi}_2$ is sufficient to completely nullify the effect of ξ_2^2 where the converse is not necessarily the case.

5. CONCLUSION

From the above discussions, we note that while neglecting the imperfection parameters $\bar{\xi}_1$ and $\bar{\xi}_2$ automatically implies, among other things, neglecting the effects of the non-linear terms $k_1\xi_1^2$ and $k_2\xi_2^2$ respectively. Also, we observe that the only condition under which the effect of the coupling term $\xi_1\xi_2$ would be felt, is when the imperfection parameters $\bar{\xi}_1$ and $\bar{\xi}_2$ are not equal to zero. Moreover, our results confirm those obtained by $[21]$. Finally, we notice that we can determine the value of the dynamic buckling load λ_D for whatever number of modal imperfections.

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