Solitary Wave Solutions to the Strain Wave Equation in Microstructured Solids through the Modified Simple Equation Method

Abstract: The modified simple equation (MSE) method is an effective method in investigating exact solitary wave solutions to nonlinear evolution equations (NLEEs) in the field of applied mathematics, mathematical physics and engineering. But, when the balance number is greater than one, typically the method does not give solution. In this article, we find out a procedure to examine the exact traveling wave solutions to the strain wave equation in microstructured solids whose balance number is two. By means of this scheme, we found some new traveling wave solutions of the above mentioned equation. When the parameters receive particular values, solitary wave solutions are originated from the exact solutions. These solutions play very important role in the engineering field. We analyze and illustrate the solitary wave properties of the solutions by graph. This shows the validity, usefulness, and necessity of the MSE method.

Keywords: MSE method; nonlinear evolution equations; solitary wave solutions; exact solutions; strain wave equation; microstructured solids.

Mathematics Subject Classification: 35C07, 35C08, 35P99

1. Introduction

Physical systems are in general explained with nonlinear partial differential equations. The mathematical modeling of microstructured solid materials that change over time depends closely on the study of a variety of systems of ordinary and partial differential equations. Similar models are developed in diverse fields of study, ranging from the natural and physical sciences, population ecology to economics, infectious disease epidemiology, neural networks, biology, mechanics etc. In spite of the eclectic nature of the fields wherein these models are formulated, different groups of them contribute adequate common attributes that make it possible to examine them within a unified theoretical structure. Such study is an area of functional analysis usually called the theory of evolution equations. Therefore, the investigation of solutions to nonlinear evolution equations (NLEES) plays a very important role to uncover the obscurity of many phenomena and processes throughout the natural sciences. However, one of the essential problems is to obtain their exact solutions. Therefore, in order to find out exact solutions to NLEEs different groups of mathematicians, physicist, and engineers have been working tirelessly. Accordingly, in the recent years, they establish several methods to search exact solutions, for instance, the Darboux transformation method [1], the Jacobi elliptic function method [2, 3], the He's homotopy perturbation method [4, 5], the tanh-function method [6, 7], the extended tanh-function method [8, 9], the Lie group symmetry method [10], the variational iteration method [11], the Hirota's bilinear method [12], the Backlund transformation method [13, 14], the inverse scattering transformation method [15], the sine-cosine method [16, 17], the Painleve expansion method [18], the Adomian decomposition method [19, 20], the (G'/G)-expansion method [21-26], the first integration method [27], the F-expansion method [28], the auxiliary equation method [29], the ansatz method [30, 31], the Exp-function method [32, 33], the homogeneous balance method [34], the modified simple equation method [35-39], the $\exp(-\varphi(\eta))$-expansion method [40, 41], the Miura transformation method [42], and others.

Microstructured materials like crystallites, alloys, ceramics, and functionally graded materials have gained broad application. The modeling of wave propagation in such materials should be able to account for various scales of microstructure [43]. In the past years, many authors have studied the strain wave equation in microstructured solids, such as, Alam et al. [43] solved the strain wave equation in microstructured solids by using the new generalized (G'/G)-expansion method. Pastrone et al. [44], Porubov and Pastrone [45] examined bell-shaped and kink-shaped solutions of this engineering problem. Akbar et al. [46] constructed traveling wave solutions of this equation by using the generalized and improved (G'/G)-expansion method. The above analysis shows that several methods to achieve exact solutions to the strain wave equation in microstructured solids have been accomplished in the recent years, but the equation has not been studied by means of the MSE method. The modified simple equation method [35-39] is a lately developed augmentative method. Its computation is straightforward, systematic, and no need the computer algebra to control the supplementary equations. In this article, we have explored exact solutions including solitary wave solutions to this equation by using the MSE method.

The rest of this article is organized as follows: In section 2, we summarize the description of the method. In section 3, we employ the method to NLEEs with balance number two and in section 4, conclusions are given.

2. The Method

Let us consider the nonlinear evolution equation of the form

$$H(u, u_t, u_x, u_y, u_z, u_{xx}, u_{tt}, \ldots) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, H is a polynomial in $u(x,t)$ and its partial derivatives, which include the highest order derivatives and nonlinear terms of the highest order, and the subscripts denote partial derivatives. In order to solve (2.1) by means of the MES method [35-39], we have to execute the following steps:

Step 1: The traveling wave variable,

$$u(x, y, z, t) = u(\xi), \qquad \xi = k(x + y + z \pm \omega\, t) \tag{2.2}$$

allows us to change the Eq. (2.1) into the following ordinary differential equation (ODE):

$$F(u, u', u'', \cdots) = 0, \tag{2.3}$$

where F is a polynomial in $u(\xi)$ and its derivatives, wherein $u'(\xi) = \dfrac{du}{d\xi}$.

Step 2: We suppose that Eq. (2.3) has the solution in the form,

$$u(\xi) = \sum_{i=0}^{N} a_i \left[\frac{s'(\xi)}{s(\xi)} \right]^i, \tag{2.4}$$

where a_i, $(i = 0,1,2,\cdots,N)$ are constants to be determined, such that $a_N \neq 0$, and $s(\xi)$ is an unidentified function to be evaluated. In sine-cosine method, tanh-function method, (G'/G)-expansion method, Jacobi elliptic function method, Exp-function method etc., the solutions are proposed in terms of some functions established in advance. But, in the MSE method, $s(\xi)$ is neither pre-defined nor a solution of any prescribed differential equation. Therefore, it is not possible to conjecture from earlier what kind of solutions one may get through this method. This is the individuality and distinction of this method. Therefore, some new solutions might be found by this method.

Step 3: The positive integer N appearing in Eq. (2.4) can be determined by taking into account the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order occurring in Eq. (2.3). If the degree of $u(\xi)$ is $\deg[u(\xi)] = N$, then the degree of the other expressions will be as follows:

$$\deg[\frac{d^m u(\xi)}{d\xi^m}] = N + m, \ \deg[u^m (\frac{d^l u(\xi)}{d\xi^l})^p] = mN + p(N + l).$$

Step 4: We substitute (2.4) into (2.3) and then we account the function $s(\xi)$. As a result of this substitution, we get a polynomial of $(s'(\xi)/s(\xi))$ and its derivatives. In the resultant polynomial, we equate the coefficients of $(s(\xi))^{-i}$, $(i = 0,1,2,...,N)$ to zero. This procedure yields a system of algebraic and differential equations which can be solved for getting a_i $(i = 0,1,2,\cdots,N)$, $s(\xi)$ and the value of the other needful parameters.

3. Application of the Method

In this section, we will execute the application of the MSE method to extract solitary wave solutions to the strain wave equation in microstructured solids which is a very important equation in the field of engineering. Let us consider the strain wave equation in microstructured solids:

$$u_{tt} - u_{xx} - \varepsilon\alpha_1 (u^2)_{xx} - \gamma\alpha_2 u_{xxt} + \delta\alpha_3 u_{xxxx} - (\delta\alpha_4 - \gamma^2\alpha_7) u_{xxtt} + \gamma\delta(\alpha_5 u_{xxxxt} + \alpha_6 u_{xxttt}) = 0 \,. \quad (3.1)$$

3.1. The Non-dissipative Case: The system is non-dissipative, if $\gamma = 0$ and governed by the double dispersive equation (see [43], [48] for details)

$$u_{tt} - u_{xx} - \varepsilon\alpha_1 (u^2)_{xx} + \delta\alpha_3 u_{xxxx} - \delta\alpha_4 u_{xxtt} = 0. \quad (3.2)$$

The balance between dispersion and nonlinearities happen when $\delta = O(\varepsilon)$. Therefore, (3.2) becomes

$$u_{tt} - u_{xx} - \varepsilon\{\alpha_1 (u^2)_{xx} - \alpha_3 u_{xxxx} + \alpha_4 u_{xxtt}\} = 0. \quad (3.3)$$

In order to extract solitary wave solutions of the strain wave equation in microstructured solids by using the MSE method, we use the wave variable

$$u(x,t) = U(\xi),\ \xi = k(x - \omega\, t)\,. \quad (3.4)$$

The wave transformation (3.4) reduces Eq. (3.3) into the ODE in the following form:

$$(\omega^2 - 1)U'' - \varepsilon\,\alpha_1 (U^2)'' + \varepsilon\, k^2 (\alpha_3 - \omega^2 \alpha_4) U^{(4)} = 0\,. \quad (3.5)$$

where prime denotes derivatives with respect to ξ. Now, integrating Eq. (3.5) twice with respect to ξ, we get a new ODE in the form:

$$(\omega^2 - 1)U - \varepsilon\,\alpha_1 U^2 + \varepsilon\, k^2 (\alpha_3 - \omega^2\alpha_4) U'' = 0\,, \quad (3.6)$$

where the integration constants are set zero, as we are seeking solitary wave solutions. Balancing the highest order derivative term U'' and the nonlinear term of the highest order U^2 involving in Eq. (3.6), we get $N = 2$. Thus, the solution (2.4) becomes

$$U(\xi) = a_0 + a_1\left(\frac{s'}{s}\right) + a_2\left(\frac{s'}{s}\right)^2, \quad (3.7)$$

where a_0, a_1 and a_2 are constants to be determined later such that $a_2 \neq 0$, and $s(\xi)$ is an unknown function. The derivatives of U are given in the following:

$$U' = -\frac{a_1 (s')^2}{s^2} - \frac{2 a_2 (s')^3}{s^3} + \frac{a_1 s''}{s} + \frac{2 a_2 s' s''}{s^2}. \tag{3.8}$$

$$U'' = \frac{2 a_1 (s')^3}{s^3} + \frac{6 a_2 (s')^4}{s^4} - \frac{3 a_1 s' s''}{s^2} - \frac{10 a_2 (s')^2 s''}{s^3} + \frac{2 a_2 (s'')^2}{s^2} + \frac{a_1 s'''}{s} + \frac{2 a_2 s' s'''}{s^2}. \tag{3.9}$$

Substituting the values of U, U' and U'' into Eq. (3.6) and then equating the coefficients of $s^j,\ j = 0,1,2,\cdots$ to zero, we respectively obtain

$$a_0(-1 + \omega^2 - \varepsilon a_0 \alpha_1) = 0. \tag{3.10}$$

$$a_1 \left((-1 + \omega^2 - 2\varepsilon a_0 \alpha_1) s' + k^2 \varepsilon (\alpha_3 - \omega^2 \alpha_4) s''' \right) = 0. \tag{3.11}$$

$$\begin{aligned} &-\varepsilon a_1 s' (a_1 \alpha_1 s' + 3k^2(\alpha_3 - \omega^2 \alpha_4) s'') + a_2 (-1 + \omega^2 - 2\varepsilon a_0 \alpha_1)(s')^2 \\ &+ a_2 (2k^2 \varepsilon (\alpha_3 - \omega^2 \alpha_4)(s'')^2) + 2k^2 \varepsilon (\alpha_3 - \omega^2 \alpha_4) s' s'')) = 0. \end{aligned} \tag{3.12}$$

$$-2\varepsilon (s')^2 \left(a_1 (a_2 \alpha_1 + k^2(-\alpha_3 + \omega^2 \alpha_4)) s' + 5k^2 a_2 (\alpha_3 - \omega^2 \alpha_4) s'' \right) = 0. \tag{3.13}$$

$$-\varepsilon a_2 \left(a_2 \alpha_1 - 6k^2 (\alpha_3 - \omega^2 \alpha_4) \right) (s')^4 = 0. \tag{3.14}$$

From Eq. (3.10), we obtain

$$a_0 = 0,\ \frac{-1 + \omega^2}{\varepsilon \alpha_1}.$$

And Eq. (3.14), yields

$$a_2 = \frac{6(k^2 \alpha_3 - k^2 \omega^2 \alpha_4)}{\alpha_1}, \qquad \text{since } a_2 \neq 0.$$

Therefore, for the values of a_0, there arise the following cases:

Case 1: When $a_0 = 0$, from Eqs. (3.11)-(3.13), we obtain

$$a_1 = \pm \frac{6k\sqrt{(1 - \omega^2)(\alpha_3 - \omega \alpha_4)}}{\sqrt{\varepsilon \alpha_1}},$$

And

$$s(\xi)=c_2 \mp \frac{k\,c_1\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}}{\sqrt{1-\omega^2}}\,e^{\mp\frac{\xi\sqrt{1-\omega^2}}{k\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}}},$$

where c_1 and c_2 are integration constants.

Substituting the values of a_0, a_1, a_2 and $s(\xi)$ into Eq. (3.7), we obtain the following exponential form solution:

$$U(\xi)=\pm\frac{6c_1 c_2 k(1-\omega^2)^{\frac{3}{2}}\sqrt{(\alpha_3-\omega^2\alpha_4)}\,e^{\mp\frac{\xi\sqrt{1-\omega^2}}{k\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}}}}{\alpha_1\sqrt{\varepsilon}\left(c_2\sqrt{(1-\omega^2)}\,e^{\mp\frac{\xi\sqrt{1-\omega^2}}{k\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}}}-k\,c_1\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}\right)^2} \tag{3.15}$$

Simplifying the required solution (3.15), we derive the following close-form solution of the strain wave equation in microstructured solids (3.3):

$$u(x,t)=\pm\frac{6k\,c_1 c_2(1-\omega^2)\sqrt{1-\omega^2}\,\sqrt{(\alpha_3-\omega^2\alpha_4)}}{\left\{\begin{array}{l}\alpha_1\sqrt{\varepsilon}\left\{c_2\sqrt{1-\omega^2}\left(\cosh(\beta(x-\omega t))\pm\sinh(\beta(x-\omega t))\right)\right.\\ \left.+k\,c_1\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}\left(\sinh(\beta(x-\omega t))\mp\cosh(\beta(x-\omega t))\right)\right\}^2\end{array}\right\}}, \tag{3.16}$$

where $\beta=\dfrac{\sqrt{1-\omega^2}}{2\sqrt{\varepsilon(\alpha_2-\omega^2\alpha_4)}}$. Solution (3.16) is the generalized solitary wave solution of the strain wave equation in microstructured solids. Since c_1 and c_2 are arbitrary constants, one might arbitrarily choose their values. Therefore, if we choose $c_1=\sqrt{1-\omega^2}$ and $c_2=\mp k\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}$ then from (3.16), we obtain the following bell shaped soliton solution:

$$u_1(x,t)=\frac{-3}{2\varepsilon\alpha_1}(1-\omega^2)\,\mathrm{sec}\,h^2\left(\frac{\sqrt{(1-\omega^2)}\,(x-\omega t)}{2\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}}\right). \tag{3.17}$$

Again, if we choose $c_1=\sqrt{1-\omega^2}$ and $c_2=\pm k\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}$, then from (3.16), we obtain the following singular soliton:

$$u_2(x,t)=\frac{3}{2\,\varepsilon\,\alpha_1}(1-\omega^2)\operatorname{csc}h^2\left(\frac{\sqrt{(1-\omega^2)}\,(x-\omega t)}{2\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}\right). \tag{3.18}$$

On the other hand, when $c_1=\sqrt{1-\omega^2}$ and $c_2=i\,k\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}$, from solution (3.16), we derive the solitary wave solutions in the form:

$$u_3(x,t)=\frac{-3(1-\omega^2)}{\varepsilon\,\alpha_1\left\{1-i\sinh\left(\frac{\sqrt{(1-\omega^2)}\,(x-\omega t)}{\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}\right)\right\}}\,. \tag{3.19}$$

Again when $c_1=\sqrt{1-\omega^2}$ and $c_2=-\,i\,k\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}$, then the generalized solitary wave solution (3.16) can be simplified as:

$$u_4(x,t)=\frac{-3(1-\omega^2)}{\varepsilon\,\alpha_1\left\{1+i\sinh\left(\frac{\sqrt{(1-\omega^2)}\,(x-\omega t)}{\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}\right)\right\}}\,. \tag{3.20}$$

The other choices of c_1 and c_2, we might obtain much new and more general exact solutions of Eq. (3.3) by the MSE method. For succinctness, other solutions have been omitted.

Case 2: When $a_0=\dfrac{-1+\omega^2}{\varepsilon\,\alpha_1}$, then Eqs. (3.11)-(3.13) yield

$$a_1=\pm\frac{6\,k\sqrt{(-1+\omega^2)(\alpha_3-\omega^2\alpha_4)}}{\alpha_1\sqrt{\varepsilon}}$$

And

$$s(\xi)=c_2\mp\frac{k\,c_1\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}{\sqrt{-1+\omega^2}}\,e^{\mp\frac{\xi\sqrt{-1+\omega^2}}{k\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}},$$

where c_1 and c_2 are constants of integration.

Now, by means of the values of a_0, a_1, a_2 and $s(\xi)$, from Eq. (3.7), we obtain the subsequent solution:

$$U(\xi)=\frac{\left\{\begin{array}{l}(-1+\omega^2)\left((-1+\omega^2)\,c_2^2\,e^{\pm 2\gamma\xi}+k^2c_1^2\,\varepsilon\,(\alpha_3-\omega^2\alpha_4)\right.\\ \left.\pm 4\,k\,c_1c_2\sqrt{\varepsilon\,(-1+\omega^2)(\alpha_3-\omega^2\alpha_4)}\,e^{\pm\gamma\xi}\right)\end{array}\right\}}{\varepsilon\,\alpha_1\left(\sqrt{(-1+\omega^2)}\,c_2\,e^{\pm\gamma\xi}\mp k\,c_1\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}\right)^2}, \tag{3.21}$$

where $\gamma=\dfrac{\sqrt{-1+\omega^2}}{k\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}$. Now, transforming the required exponential function solution (3.21) into trigonometric function, we obtain the following solution to the strain wave equation in the microstructured solids:

$$u(x,t)=\frac{\left\{\begin{array}{l}(-1+\omega^2)\left(\pm 4\,k\,c_1\,c_2\sqrt{\varepsilon\,(-1+\omega^2)(\alpha_3-\omega^2\,\alpha_4)}\right.\\ \left.+c_2^2(-1+\omega^2)(\cosh\theta\pm\sinh\theta)+\varepsilon\,c_1^2\,k^2(\alpha_3-\omega^2\,\alpha_4)(\cosh\theta\mp\sinh\theta)\right)\end{array}\right\}}{\left\{\begin{array}{l}\varepsilon\,\alpha_1\left(\sqrt{\varepsilon\,(-1+\omega^2)}\,(\cosh(\theta/2)\pm\sinh(\theta/2))c_2\right.\\ \left.+k\sqrt{\varepsilon\,(\alpha_3-\omega^2\,\alpha_4)}\,(\sinh(\theta/2)\mp\cosh(\theta/2))\right)^2\end{array}\right\}} \tag{3.22}$$

Thus, we acquire the generalized solitary wave solution (3.22) to the strain wave equation in microstructured solids, where $\theta=\dfrac{(x-\omega t)\sqrt{-1+\omega^2}}{\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}$. Since c_1 and c_2 are integration constants, therefore, somebody might randomly pick their values. So, if we pick $c_1=\sqrt{\omega^2-1}$ and $c_2=\pm k\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}$, then the solitary wave solution (3.22) reduces to:

$$u_5(x,t)=\frac{\omega^2-1}{2\,\varepsilon\,\alpha_1}\left\{2+\cosh\left(\frac{\sqrt{(\omega^2-1)}\,(x-\omega t)}{\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}\right)\right\}\operatorname{csc}h^2\left(\frac{\sqrt{(\omega^2-1)}\,(x-\omega t)}{2\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}\right) \tag{3.23}$$

Again, if we pick $c_1=\sqrt{\omega^2-1}$ and $c_2=\mp k\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}$, then from (3.22), we obtain the subsequent solitary wave solution:

$$u_6(x,t)=\frac{\omega^2-1}{2\,\varepsilon\,\alpha_1}\left\{-2+\cosh\left(\frac{\sqrt{(\omega^2-1)}\,(x-\omega t)}{\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}\right)\right\}\operatorname{sec}h^2\left(\frac{\sqrt{(\omega^2-1)}\,(x-\omega t)}{2\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}}\right) \tag{3.24}$$

Moreover, if we pick $c_1=\sqrt{\omega^2-1}$ and $c_2=i\,k\sqrt{\varepsilon\,(\alpha_3-\omega^2\alpha_4)}$, then from (3.22), we obtain the following solution:

$$u_7(x,t)=\frac{(\omega^2-1)\left(-2i+\sinh\left(\frac{\sqrt{(\omega^2-1)}\,(x-\omega t)}{\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}}\right)\right)}{\varepsilon\alpha_1\left(i+\sinh\left(\frac{\sqrt{(\omega^2-1)}\,(x-\omega t)}{\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}}\right)\right)}. \tag{3.25}$$

Again, if we pick $c_1=\sqrt{\omega^2-1}$ and $c_2=-ik\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}$, then from (3.22), we derive the following solution:

$$u_8(x,t)=\frac{(\omega^2-1)\left(2i+\sinh\left(\frac{\sqrt{(\omega^2-1)}\,(x-\omega t)}{\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}}\right)\right)}{\varepsilon\alpha_1\left(-i+\sinh\left(\frac{\sqrt{(\omega^2-1)}\,(x-\omega t)}{\sqrt{\varepsilon(\alpha_3-\omega^2\alpha_4)}}\right)\right)}. \tag{3.26}$$

Since c_1 and c_2 are arbitrary constants for other choices of c_1 and c_2, we might obtain much new and more general exact solutions of Eq. (3.3) by the MSE method without any aid of symbolic computation software. The major advantage of the MSE method is that the calculations are not sophisticated and easy to control. It is not required any computer algebra system to facilitate the calculations, whereas to the Exp-function method, the (G'/G)-expansion, the tanh-function method, the homotopy analysis method etc. the computer algebra system is very much needed. However, the solutions obtained by the MSE method are equivalent to those solutions obtained by the above mentioned method.

Remark 1: Solutions (3.17)-(3.20) and (3.23)-(3.26) have been verified by putting them back into the original equation and found correct.

3.2. The Dissipative Case: If $\gamma\neq 0$, then the system is dissipative. Therefore, for $\delta=\gamma=O(\varepsilon)$, the balance should be between nonlinearity, dispersion and dissipation, perturbed by the higher order dissipative terms to the strain wave equation in microstructured solids,

$$u_{tt} - u_{xx} - \varepsilon\left(\alpha_1 (u^2)_{xx} + \alpha_2 u_{xxt} - \alpha_3 u_{xxxx} + \alpha_4 u_{xxtt}\right) = 0 . \tag{3.27}$$

since $\varepsilon \to 0,$ so the higher order term are omitted.

The traveling wave transformation (3.4) reduces Eq. (3.27) to the following ODE:

$$(\omega^2 - 1)U'' - \varepsilon(\alpha_1 (U^2)'' - k\,\omega\,\alpha_2\, U''' - k^2(\alpha_3 - \omega^2 \alpha_4)U^{(4)} = 0 . \tag{3.28}$$

where prime denotes the derivatives with respect to ξ. Integrating Eq. (3.28) with respect to ξ, we obtain

$$(\omega^2 - 1)U - \varepsilon\,\alpha_1\, U^2 + \varepsilon\, k\,\omega\,\alpha_2\, U' + \varepsilon\, k^2(\alpha_3 - \omega^2 \alpha_4)U'' = 0 . \tag{3.29}$$

Balancing the highest order linear term and the nonlinear terms of the highest order, we get $N = 2$. Therefore, the shape of the solution of Eq. (3.29) is one and the same to the form of the solution (3.7).

Substituting the values of U, U' and U'' into Eq. (3.29) and then equating the coefficients of s^0, s^{-1}, s^{-2}, s^{-3}, s^{-4} to zero, we respectively obtain

$$a_0(-1 + \omega^2 - \varepsilon\, a_0\, \alpha_1) = 0. \tag{3.30}$$

$$a_1\left((-1 + \omega^2 - 2\,\varepsilon\, a_0\, \alpha_1)\, s' + k\,\varepsilon\,\omega\,\alpha_2\, s'' + k\,(\alpha_3 - \omega^2 \alpha_4)\, s'''\right) = 0 . \tag{3.31}$$

$$\begin{aligned} &-\varepsilon\, a_1\, s'\left(a_1\,\alpha_1\, s' + k\,(\omega\,\alpha_2\, s' + 3\,k(\alpha_3 - \omega^2\alpha_4)\, s'')\right) + a_2\,((-1 + \omega^2 - 2\,\varepsilon\, a_0 \alpha_1)\,(s')^2 \\ &+ 2\,k^2 \varepsilon\,(\alpha_3 - \omega^2\alpha_4)(s'')^2) + 2\,a_2 k\,\varepsilon\, s'\,(\omega\,\alpha_2\, s'' + k\,(\alpha_3 - \omega^2\alpha_4)\, s''') = 0. \end{aligned} \tag{3.32}$$

$$-2\,\varepsilon\, a_1\left(a_2\,\alpha_1 + k^2(-\alpha_3 + \omega^2\alpha_4)\right)(s')^3 - 2\,\varepsilon\, k\, a_2\,(s')^2\left(\omega\,\alpha_2 s' + 5\,k\,(\alpha_3 - \omega^2\alpha_4)\, s''\right) = 0. \tag{3.33}$$

$$-\varepsilon\, a_2\left(a_2\alpha_1 - 6\,k^2(\alpha_3 - \omega^2\alpha_4)\right)(s')^4 = 0. \tag{3.34}$$

From Eqs. (3.30) and (3.34), we obtain

$$a_0 = 0, \frac{-1 + \omega^2}{\varepsilon\,\alpha_1} \text{ and } a_2 = \frac{6k^2(\alpha_3 - \omega^2\alpha_4)}{\alpha_1}, \text{ since } a_2 \neq 0 .$$

Therefore, depending on the values of a_0, the following different cases arise:

Case 1: When $a_0 = 0$, from Eqs. (3.31) - (3.33), we get

$$a_1 = 0,$$

$$\omega = \begin{cases} \pm \dfrac{1}{5\sqrt{2}} \sqrt{25 - \dfrac{6\varepsilon \alpha_2^2}{\alpha_4} + \dfrac{25\alpha_3}{\alpha_4} - \dfrac{\sqrt{(6\varepsilon\alpha_2^2 - 25\alpha_3 - 25\alpha_4)^2 - 2500\,\alpha_3\alpha_4}}{\alpha_4}} = \pm\mu_1 \\ \pm \sqrt{\dfrac{1}{2} - \dfrac{3\varepsilon\alpha_2^2}{25\alpha_4} + \dfrac{\alpha_3}{2\alpha_4} + \dfrac{\sqrt{(6\varepsilon\alpha_2^2 - 25\alpha_3 - 25\alpha_4)^2 - 2500\,\alpha_3\alpha_4}}{50\,\alpha_4}} = \pm\mu_2 \end{cases} \quad \text{(say)}$$

And

$$s(\xi) = c_2 \pm \frac{c_1}{h} e^{\pm h\xi}$$

where c_1 and c_2 are integration constants and $h = -\dfrac{\omega\alpha_2}{5k\left(\alpha_3 - \omega^2\alpha_4\right)}$.

Now, substituting the values of a_0, a_1, a_2 and $s(\xi)$ into Eq. (3.7), we obtain the following solution:

$$U(\xi) = \frac{6k^2c_1^2(\alpha_3 - \omega^2\alpha_4)}{\alpha_1\left(\dfrac{c_1}{h}e^{h\xi} + c_2\right)^2} e^{2h\xi}. \tag{3.35}$$

Simplifying the required solution (3.35), we derive the following close-form solution of the strain wave equation in microstructured solids (3.27):

$$u(x,t) = \frac{6h^2k^2c_1^2(\alpha_3 - \omega^2\alpha_4)\left\{\cosh\left(hk(x-\omega t)\right) + \sinh\left(hk(x-\omega t)\right)\right\}}{\alpha_1\left\{\left(\cosh\left(hk(x-\omega t)/2\right) + \sinh\left(hk(x-\omega t)/2\right)\right)c_1 + h\left(\cosh\left(hk(x-\omega t)/2\right) - \sinh\left(hk(x-\omega t)/2\right)\right)c_2\right\}^2}. \tag{3.36}$$

Since c_1 and c_2 are integration constants, one might arbitrarily select their values. If we choose $c_1 = h$ and $c_2 = 1$, then from (3.36), we obtain

$$u_9(x,t) = \frac{3h^2k^2(\alpha_3 - \omega^2\alpha_4)}{2\alpha_1}\left(\cosh(hk(x-\omega t)) + \sinh(hk(x-\omega t))\right)\operatorname{sech}^2\left(\frac{1}{2}hk(x-\omega t)\right) \tag{3.37}$$

Again if we choose $c_1 = h$ and $c_2 = -1$, then from (3.36), we obtain the following solitary wave solution:

$$u_{10}(x,t) = \frac{3h^2k^2(\alpha_3 - \omega^2\alpha_4)}{2\alpha_1}\big(\cosh(hk(x-\omega t)) + \sinh(hk(x-\omega t))\big)\operatorname{csc}h^2\left(\frac{1}{2}hk(x-\omega t)\right) \tag{3.38}$$

Case 2: When $a_0 = \dfrac{-1+\omega^2}{\varepsilon\alpha_1}$, from Eq.(3.31) - (3.33), we obtain

$$a_1 = 0,$$

$$\omega = \begin{cases} \pm\dfrac{1}{5\sqrt{2}}\sqrt{\dfrac{6\varepsilon\alpha_2^2 + 25(\alpha_3+\alpha_4) - \sqrt{(6\varepsilon\alpha_2^2 + 25(\alpha_3+\alpha_4))^2 - 2500\alpha_3\alpha_4}}{\alpha_4}} = \pm\theta_1 \\ \pm\dfrac{1}{5\sqrt{2}}\sqrt{\dfrac{6\varepsilon\alpha_2^2 + 25(\alpha_3+\alpha_4) + \sqrt{(6\varepsilon\alpha_2^2 + 25(\alpha_3+\alpha_4))^2 - 2500\alpha_3\alpha_4}}{\alpha_4}} = \pm\theta_2 \end{cases} \quad \text{(say)}$$

And

$$s(\xi) = c_2 - \frac{5kc_1(\alpha_3 - \omega^2\alpha_4)}{\omega\alpha_2} e^{\frac{-\omega\alpha_2}{5k(\alpha_3-\omega^2\alpha_4)}\xi},$$

where c_1 and c_2 are constants of integration.

Substituting the values of a_0, a_1, a_2 and $s(\xi)$ into Eq. (3.7), we obtain the solution in the form:

$$U(\xi) = \frac{-1+\omega^2}{\varepsilon\alpha_1} + \frac{6k^2\omega^2c_1^2\alpha_2^2(\alpha_3 - \omega^2\alpha_4)}{\alpha_1\left(\omega c_2\alpha_2 e^{\frac{\omega\alpha_2}{5k(\alpha_3-\omega^2\alpha_4)}\xi} - 5kc_1(\alpha_3 - \omega^2\alpha_4)\right)^2}. \tag{3.39}$$

Simplifying the required exponential function solution (3.39) into trigonometric function solution, we derive the solution of Eq. (3.27) as follows:

$$u(x,t)=\frac{\omega^2\alpha_2^2c_2^2(\omega^2-1)\big(\cosh(\varphi)-\sinh(\varphi)\big)-10\,k\,\omega\,c_1\,c_2\,\alpha_2(\omega^2-1)(\alpha_3-\omega^2\alpha_4)+k^2c_1^2(\alpha_3-\omega^2\alpha_4)\big(6\,\varepsilon\,\omega^2\alpha_2^2+25(\omega^2-1)(\alpha_3-\omega^2\alpha_4)\big)\big(\cosh(\varphi)+\sinh(\varphi)\big)}{\varepsilon\,\alpha_1\left\{\omega\,c_2\alpha_2\left(\cosh\left(\frac{\varphi}{2}\right)-\sinh\left(\frac{\varphi}{2}\right)\right)-5\,k(\alpha_3-\omega^2\alpha_4)\,c_1\left(\cosh\left(\frac{\varphi}{2}\right)+\sinh\left(\frac{\varphi}{2}\right)\right)\right\}^2}$$

(3.40)

Therefore, we obtain the generalized soliton solution (3.40) to the strain wave equation in microstructured solids, where $\varphi=-\dfrac{\omega\,\alpha_2\,(x-\omega t)}{5(\alpha_3-\omega^2\alpha_4)}$ and $\omega=\pm\theta_1$ or $\pm\theta_2$. But, since c_1 and c_2 are arbitrary constants, someone may arbitrarily choose their values. So, if we choose $c_1=\alpha_2\,\omega$ and $c_2=5\,k\,(\alpha_3-\omega^2\alpha_4)$, the solitary wave solution (3.40) becomes

$$u_{11}(x,t)=\frac{3\,\varepsilon\,\omega^2\alpha_2^2\left(1+\coth\left[\dfrac{\omega\,\alpha_2\,(-x+\omega t)}{10(\alpha_3-\omega^2\alpha_4)}\right]\right)^2+50(\omega^2-1)(\alpha_3-\omega^2\alpha_4)}{50\,\varepsilon\,\alpha_1\,(\alpha_3-\omega^2\alpha_4)} \tag{3.41}$$

Again, if we choose $c_1=\alpha_2\,\omega$ and $c_2=-5\,k\,(\alpha_3-\omega^2\alpha_4)$, from (3.20), we obtain the following solitary wave solutions in the form:

$$u_{12}(x,t)=\frac{3\,\varepsilon\,\omega^2\alpha_2^2\left(1+\tanh\left[\dfrac{\omega\,\alpha_2\,(-x+\omega t)}{10(\alpha_3-\omega^2\alpha_4)}\right]\right)^2+50(\omega^2-1)(\alpha_3-\omega^2\alpha_4)}{50\,\varepsilon\,\alpha_1\,(\alpha_3-\omega^2\alpha_4)} \tag{3.42}$$

Since c_1 and c_2 are arbitrary constants for other choices of c_1 and c_2, we might obtain much new and more general exact solutions of Eq. (3.27) by the MSE method without any aid of symbolic computation software.

Remark 2: Solutions (3.37)-(3.38) and (3.41)-(3.42) have been verified by putting them back into the original equation and found correct.

4. Physical interpretations of the solutions

In this sub-section, we depict the graph and signify the obtained solutions to the strain wave equation in microstructured solids for non-dissipative and dissipative. The solution u_1 in (3.17) depends on the physical parameters $\alpha_1, \alpha_3, \alpha_4, \varepsilon$ and the group velocity ω. Now, we will discus all the possible physical significances for $-2 \le \alpha_1, \alpha_3, \alpha_4, \varepsilon \le 2$, and soliton exists for $|\omega|>1$ and $|\omega|<1$. For the value of parameters $\alpha_1, \alpha_3, \alpha_4, \varepsilon<0$ and $|\omega|>1$, the solution u_1 in (3.17) represents the bell shape soliton and when $|\omega|<1$ then the solution u_1 represents the dark soliton. It is shown in Fig. 1. Also if the values of the parameters are $\alpha_1>0,\ \alpha_3, \alpha_4, \varepsilon<0$ and $|\omega|>1$, then the solution u_1 represents the dark soliton and when $|\omega|<1$, then the solution u_1 represents the bell shape soliton. It is shown the Fig. 2. Again, for $\alpha_1, \alpha_3, \alpha_4<0,\ \varepsilon>0$ and $|\omega|<1$, the solution u_1 in (3.17) represents the multi-soliton and when $|\omega|>1$, the solution u_1 represents the dark soliton. It is plotted in Fig. 3. Again, if the values of the physical parameters are $\alpha_1>0,\ \alpha_3, \alpha_4<0,\ \varepsilon>0$ and $|\omega|>1$, then the solution u_1 represents the dark soliton and when $|\omega|<1$ then the solution u_1 represents the bell shape soliton. It is shown in Fig. 4. We can sketch the other figures of the solution u_1 for different values of the parameters. But for page limitation in this article we have omitted these figures. So, for other cases we do not draw the figures but we discuss for other cases with the following table:

$\varepsilon>0$	$\lvert\omega\rvert>1$	$\alpha_1<0,\ \alpha_3<0,\ \alpha_4<0$	Dark soliton
		$\alpha_1>0,\ \alpha_3<0,\ \alpha_4<0$	Bell shape soliton
		$\alpha_1>0,\ \alpha_3>0,\ \alpha_4<0$	Bell shape soliton
		$\alpha_1>0,\ \alpha_3>0,\ \alpha_4>0$	Bell shape soliton
		$\alpha_1>0,\ \alpha_3<0,\ \alpha_4>0$	Bell shape soliton
		$\alpha_1<0,\ \alpha_3>0,\ \alpha_4<0$	Dark soliton

		$\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 > 0$	Dark soliton
		$\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 > 0$	Dark soliton
	$\lvert\omega\rvert < 1$	$\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 < 0$	Bell shape soliton
		$\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 < 0$	Dark soliton
		$\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 < 0$	Dark soliton
		$\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 > 0$	Dark soliton
		$\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 > 0$	Dark soliton
		$\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 < 0$	Bell shape soliton
		$\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 > 0$	Bell shape soliton
		$\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 > 0$	Periodic bell shape soliton
$\varepsilon < 0$	$\lvert\omega\rvert > 1$	$\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 < 0$	Bell shape or Periodic bell shape soliton
		$\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 < 0$	Dark soliton or Periodic dark soliton
		$\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 < 0$	Dark soliton
		$\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 > 0$	Periodic dark soliton
		$\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 > 0$	Periodic dark soliton
		$\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 < 0$	Bell shape soliton
		$\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 > 0$	Periodic bell shape soliton
		$\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 > 0$	Periodic bell shape soliton
	$\lvert\omega\rvert < 1$	$\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 < 0$	Dark soliton or Periodic dark soliton
		$\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 < 0$	Bell shape or Periodic bell shape soliton
		$\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 < 0$	Periodic bell shape soliton
		$\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 > 0$	Bell shape or Periodic bell shape soliton
		$\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 > 0$	Bell shape soliton
		$\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 < 0$	Periodic dark soliton
		$\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 > 0$	Dark soliton or Periodic dark soliton
		$\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 > 0$	Dark soliton

Also the soliton u_2 in (3.18) depends on the parameters $\alpha_1, \alpha_3, \alpha_4, \varepsilon$ and ω. Now, we will discus all the possible physical significances for $-2 \le \alpha_1, \alpha_3, \alpha_4, \varepsilon \le 2$, and soliton exists for $\lvert\omega\rvert > 1$ and $\lvert\omega\rvert < 1$. For the value of parameters contains $\alpha_1, \alpha_3, \alpha_4, \varepsilon > 0$ and $\lvert\omega\rvert > 1$, then the solution u_2 in (3.18) represents the singular dark soliton and when $\lvert\omega\rvert < 1$

then the solution u_2 represents the singular bell shape soliton. It is shown in Fig. 5. Also, for $\alpha_1, \alpha_3, \alpha_4 < 0, \varepsilon > 0$ and $|\omega| > 1$, then the solution u_2 in (3.18) represents the periodic singular dark soliton and when $|\omega| < 1$ then the solution u_2 represents the periodic singular bell shape soliton. It is plotted of the Fig. 6. On the other hand, the solutions u_3 in (3.19) and u_4 in (3.20) exist for $(\alpha_3 - \alpha_4 \omega^2) > 0,\ \varepsilon < 0$ or $(\alpha_3 - \alpha_4 \omega^2) < 0,\ \varepsilon > 0$ when $|\omega| > 1$ or $|\omega| > 1$. For the value of the parameters are $\alpha_1 = -1.25,\ \alpha_3 = -0.1,\ \alpha_4 = -2,\ \varepsilon = -1$, when $\omega = 0.96$, the solution u_3 in (3.19) represents the dark soliton and $\alpha_1 = -1.5,\ \alpha_3 = -0.1,$ $\alpha_4 = 2,\ \varepsilon = -1$, when $\omega = 1.5$, the solution u_4 represents the periodic soliton. It is shown in Fig. 7. Again, the travelling wave solution u_5 in (3.23) represents the bell shape singular solitons for $\alpha_1 = -1 = \alpha_3,\ \ \alpha_4 = 1,\ \ \varepsilon = 0.5,\ \ \omega = -1.5$ and $\omega = 0.5$ respectively, in Fig. 8 and Fig. 9 from u_6 in (3.24) represents the bell shape soliton, when $\omega = 1.5$ and the dark soliton , when $\omega = -0.75$. In Fig. 10, we have plotted of the periodic bell shape and dark type soliton for $\alpha_1 = \alpha_3 = -1.25$, $\alpha_4 = 1,\ \ \varepsilon = 0.7,\ \ \omega = -1.2$ and $\alpha_1 = \alpha_3 = -1.25$, $\alpha_4 = 1,\ \ \varepsilon = -0.7,$ $\omega = 0.25$ respectively to the solution of u_7 in (3.25) and Fig. 11 plotted the periodic dark type and bell shape soliton for $\alpha_1 = 1.25,\ \ \alpha_3 = -1.25,\ \alpha_4 = 1,\ \ \varepsilon = 0.7,\ \ \omega = -1.2$ and $\alpha_1 = \alpha_3 = -1.25$, $\alpha_4 = 1,\ \ \varepsilon = -0.7,\ \ \omega = -0.25$ respectively to the solution of u_8 in (3.26). Fig. 12 and 13 represent the kink shape solutions u_9 given in (3.37) are respectively, for $\alpha_1 = 1,\ \ \alpha_2 = 1,\ \ \alpha_3 = -1.5,\ \ \alpha_4 = -1,\ \ \varepsilon = 0.5$ and $\alpha_1 = -1,\ \ \alpha_2 = 1,\ \ \alpha_3 = -1.5,\ \ \alpha_4 = -1,$ $\varepsilon = 0.5$ respectively, when $\omega = \pm\mu_1$ and for $\alpha_1 = 1,\ \ \alpha_2 = 1,\ \ \alpha_3 = -1.5,\ \ \alpha_4 = -1,\ \ \varepsilon = 0.5$ and $\alpha_1 = -1,\ \ \alpha_2 = 1,\ \ \alpha_3 = -1.5,\ \ \alpha_4 = -1,\ \ \varepsilon = 0.5$ respectively, when $\omega = \pm\mu_2$. Also sketch the figures 14 and 15, singular bell shape solutions u_{10} in (3.38) for $\alpha_1 = 1,\ \ \alpha_2 = 1,$ $\alpha_3 = -1.5,\ \ \alpha_4 = -1,\ \ \varepsilon = 0.5$ and $\alpha_1 = -1,\ \ \alpha_2 = 1,\ \ \alpha_3 = -1.5,\ \ \alpha_4 = -1,\ \varepsilon = 0.5$

respectively, when $\omega = \pm\mu_1$ and for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -1.5$, $\alpha_4 = -1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = -1.5$, $\alpha_4 = -1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\mu_2$. On the other hand, Fig. 16 and 17 are singular bell and dark type solitons u_{11} in (3.41) for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\theta_1$ and for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\theta_2$. Also, draw the Figures 18 and 19 are kink shape solitons u_{12} in (3.42) for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\theta_1$ and for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\theta_2$. All figures are drawn within $-10 \le x,\ t \le 10$.

We can sketch the other figures or discuss the solutions u_2 to u_{12} for different values of the parameters. But for page limitation in this article we have omitted these figures in details.

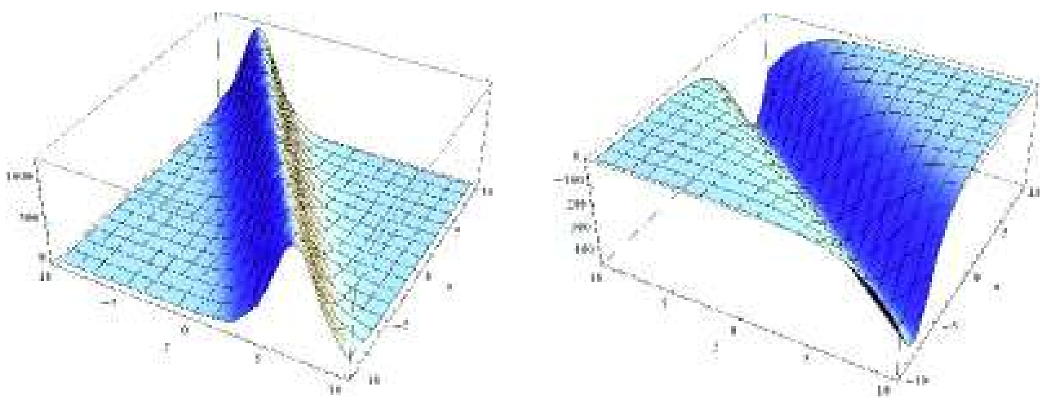

Fig. 1: Sketch of the solution u_1 in (3.17) for $\alpha_1 = -0.001$, $\alpha_3 = \alpha_4 = \varepsilon = \omega = -1.5$ and $\alpha_1 = -0.001$, $\alpha_3 = \alpha_4 = \varepsilon = \omega = -0.75$ respectively.

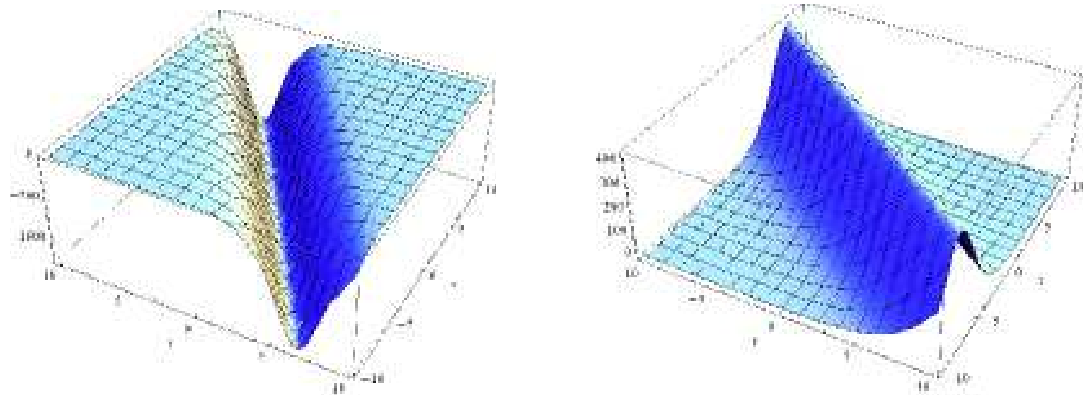

Fig. 2: Plot of the solution u_1 in (3.17) for $\alpha_1 = 0.001,\ \alpha_3 = \alpha_4 = \varepsilon = \omega = -1.5$ and $\alpha_1 = 0.001,\ \alpha_3 = \alpha_4 = \varepsilon = \omega = -0.75$ respectively.

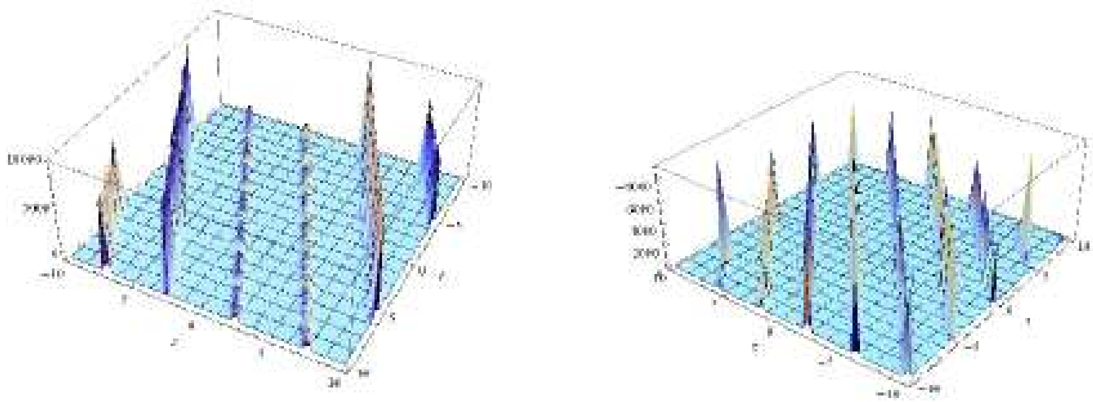

Fig. 3: Sketch of the solution u_1 in (3.17) for $\alpha_1 = \alpha_3 = \alpha_4 = -1.2,\ \varepsilon = \omega = 0.5$ and $\alpha_1 = \alpha_3 = \alpha_4 = -1.2,\ \varepsilon = 0.5,\ \omega = 1.25$ respectively.

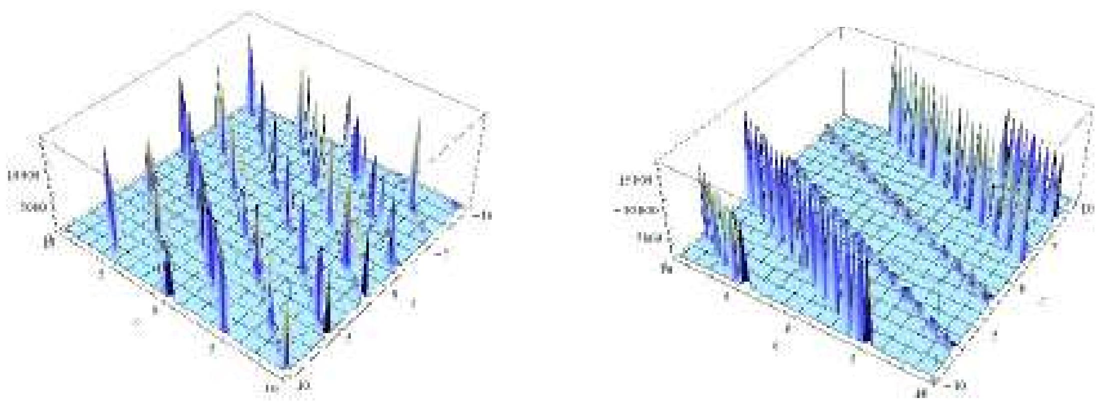

Fig. 4: Sketch of the solution u_1 in (3.17) for $\alpha_1 = 0.75,\ \alpha_3 = \alpha_4 = -1.2,\ \varepsilon = 0.5,\ \omega = 1.25$ and $\alpha_1 = 0.75,\ \alpha_3 = \alpha_4 = -1.2,\ \varepsilon = 0.5,\ \omega = 0.5$ respectively.

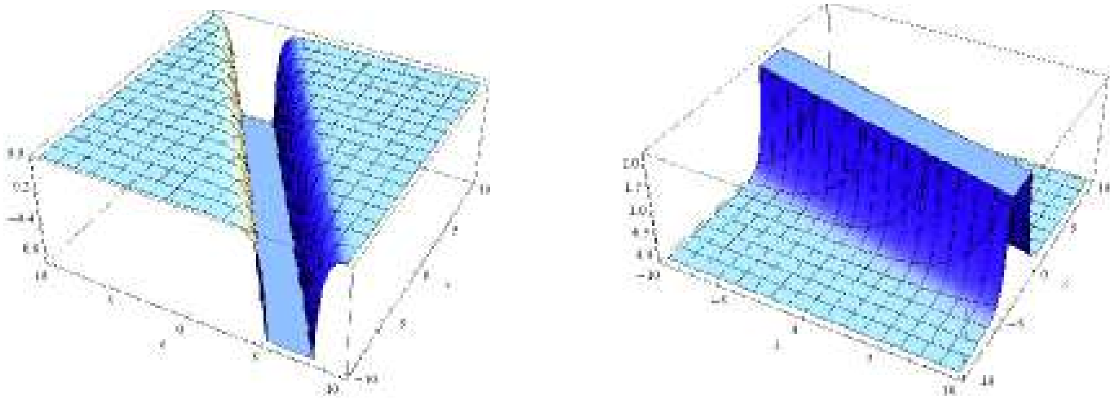

Fig. 5: Sketch of the singular dark and singular bell shape soliton u_2 in (3.18) for $\alpha_1 = \alpha_3 = \alpha_4 = 0.5$, $\varepsilon = 0.75$, $\omega = -1.5$ and $\alpha_1 = \alpha_3 = \alpha_4 = 0.5$, $\varepsilon = 0.75$, $\omega = -0.25$ respectively.

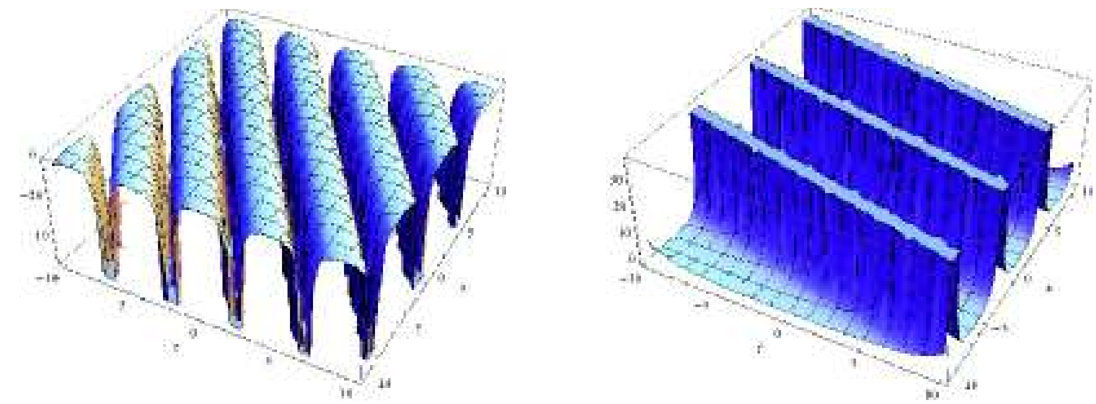

Fig. 6: Sketch of the periodic singular soliton u_2 in (3.18) for $\alpha_1 = \alpha_3 = \alpha_4 = -1.5$, $\varepsilon = 0.75$, $\omega = -1.5$ and $\alpha_1 = \alpha_3 = \alpha_4 = -1.5$, $\varepsilon = 0.75$, $\omega = -0.25$ respectively.

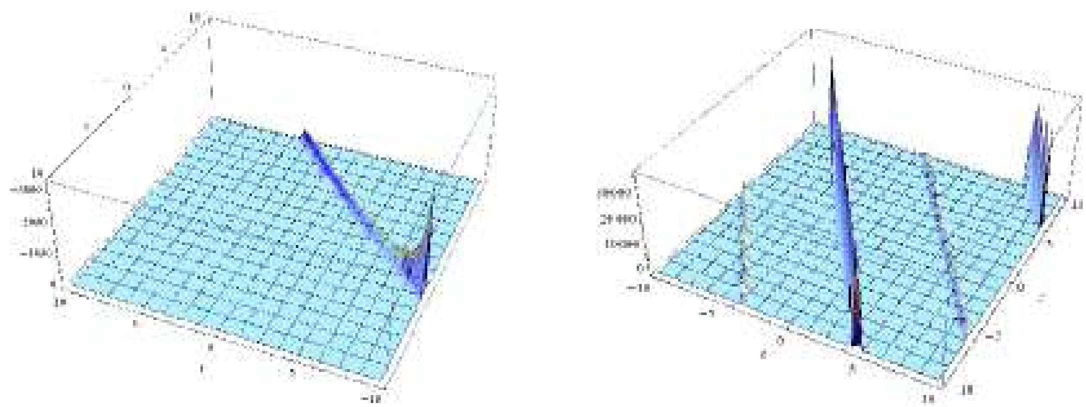

Fig. 7: Sketch of the solution u_3 in (3.19) and the solution u_4 in (3.20) for $\alpha_1 = -1.25$, $\alpha_3 = -0.1$, $\alpha_4 = -2$, $\varepsilon = -1$, $\omega = 0.96$ and $\alpha_1 = -1.5$, $\alpha_3 = -0.1$, $\alpha_4 = 2$, $\varepsilon = -1$, $\omega = 1.5$ respectively.

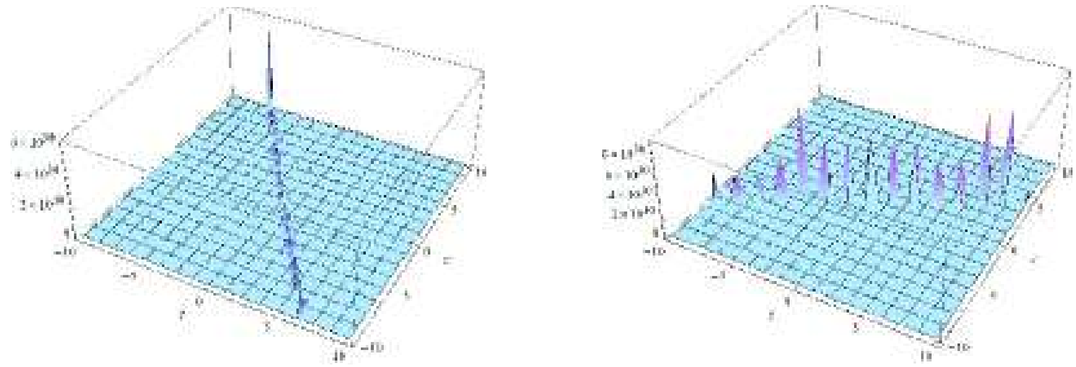

Fig. 8: Sketch of the solutions u_5 in (3.23) for $\alpha_1 = -1 = \alpha_3$, $\alpha_4 = 1$, $\varepsilon = 0.5$, $\omega = -1.5$ and $\omega = 0.5$ respectively.

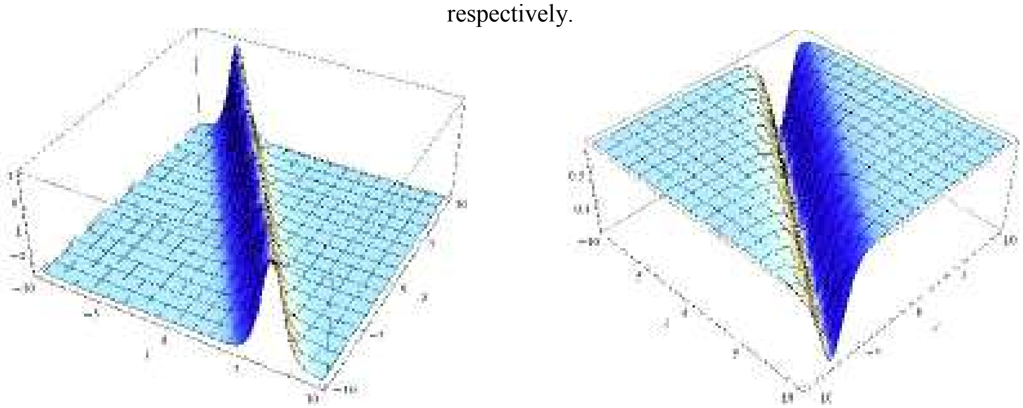

Fig. 9: Sketch of the bell shape soliton and dark soliton u_6 in (3.24) for $\alpha_1 = \alpha_3 = \alpha_4 = -1$, $\varepsilon = 0.5$, $\omega = 1.5$ and $\omega = -0.75$ respectively.

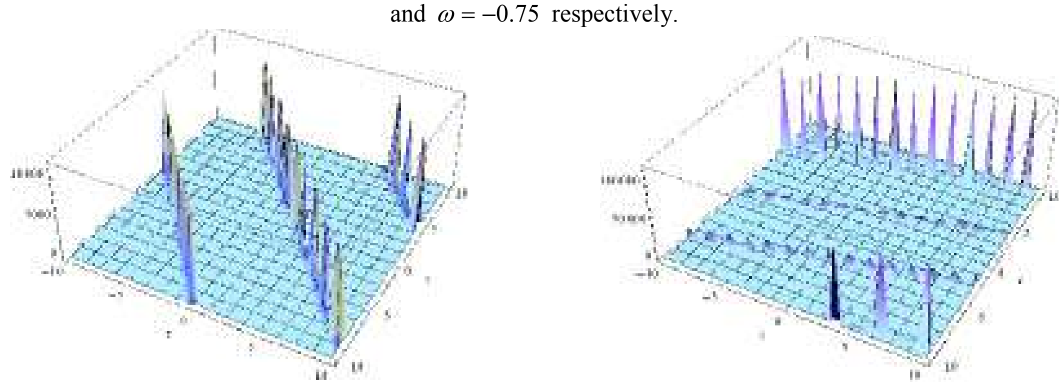

Fig. 10: Sketch of the solutions u_7 in (3.25) for $\alpha_1 = \alpha_3 = -1.25$, $\alpha_4 = 1$, $\varepsilon = 0.7$, $\omega = -1.2$ and $\alpha_1 = \alpha_3 = -1.25$, $\alpha_4 = 1$, $\varepsilon = -0.7$, $\omega = 0.25$ respectively.

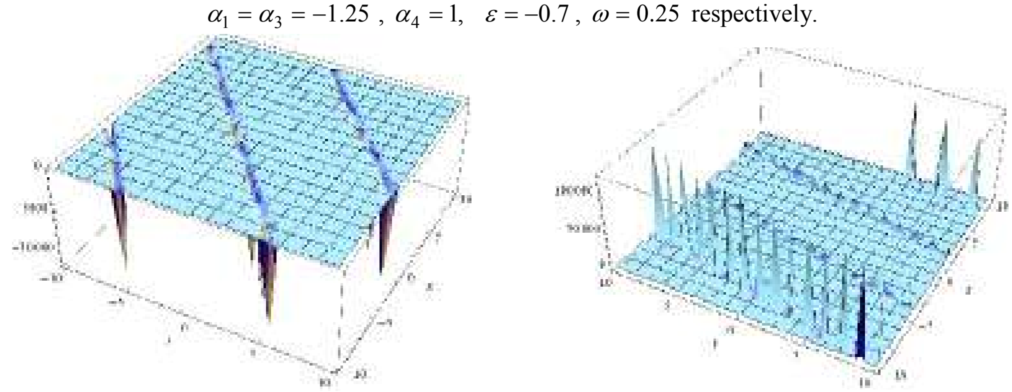

Fig. 11: Sketch of the solutions u_8 in (3.26) for $\alpha_1 = 1.25$, $\alpha_3 = -1.25$, $\alpha_4 = 1$, $\varepsilon = 0.7$, $\omega = -1.2$ and $\alpha_1 = \alpha_3 = -1.25$, $\alpha_4 = 1$, $\varepsilon = -0.7$, $\omega = -0.25$ respectively.

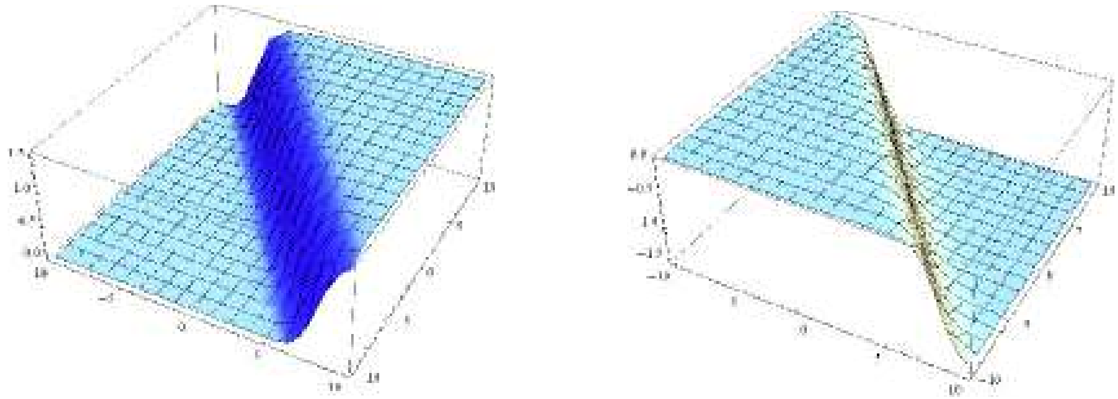

Fig. 12: Kink shape soliton obtained from u_9 in (3.37) for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -1.5$, $\alpha_4 = -1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = -1.5$, $\alpha_4 = -1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\mu_1$.

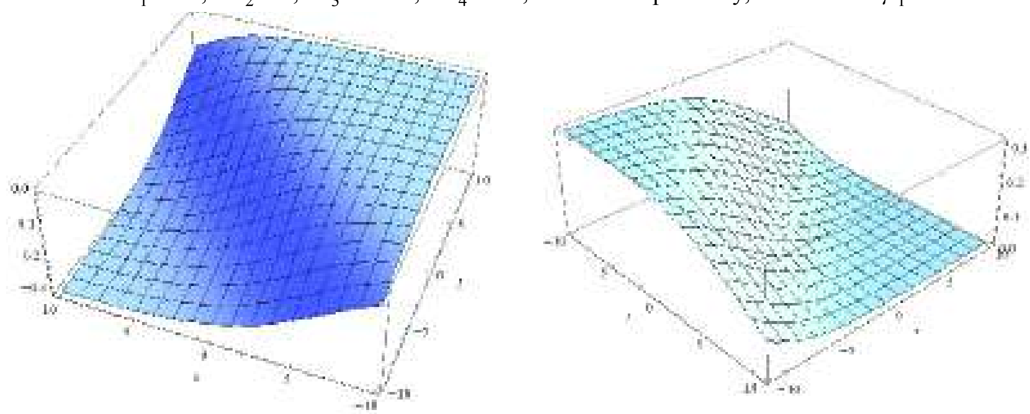

Fig. 13: Kink shape soliton obtained from u_9 in (3.37) for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -1.5$, $\alpha_4 = -1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = -1.5$, $\alpha_4 = -1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\mu_2$.

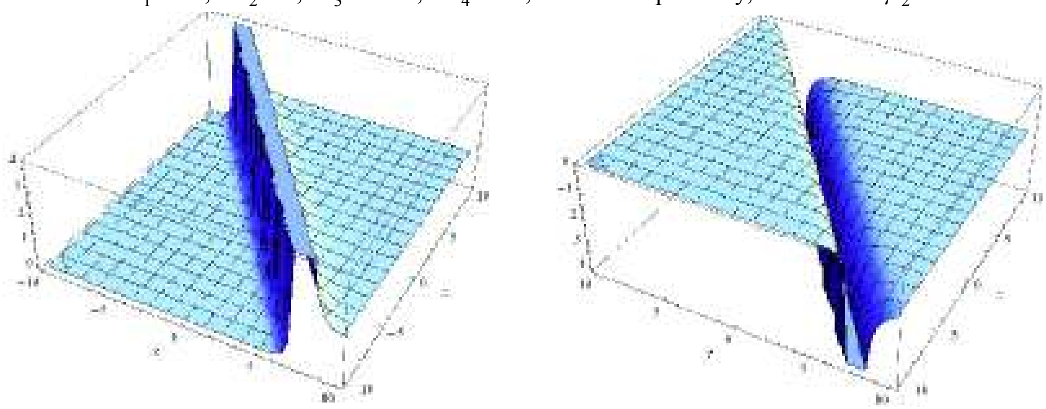

Fig. 14: Singular bell shape and anti-bell shape soliton u_{10} in (3.38) for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -1.5$, $\alpha_4 = -1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = -1.5$, $\alpha_4 = -1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\mu_1$.

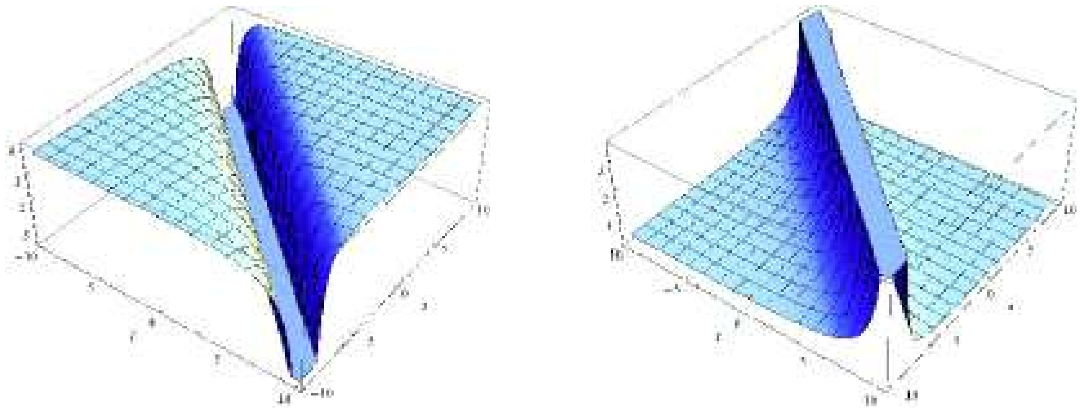

Fig. 15: Singular anti-bell shape and bell shape soliton u_{10} in (3.38) for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -1.5$, $\alpha_4 = -1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = -1.5$, $\alpha_4 = -1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\mu_2$.

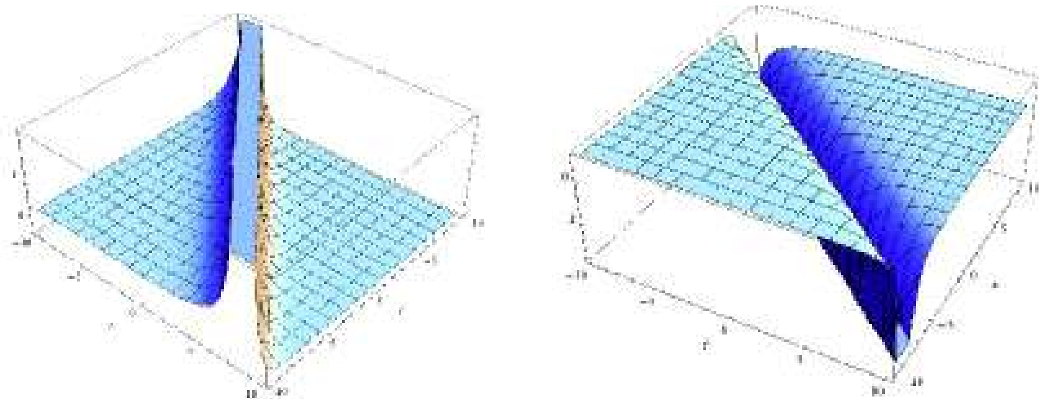

Fig. 16: Sketch the singular bell type and anti-bell soliton u_{11} in (3.41) for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\theta_1$.

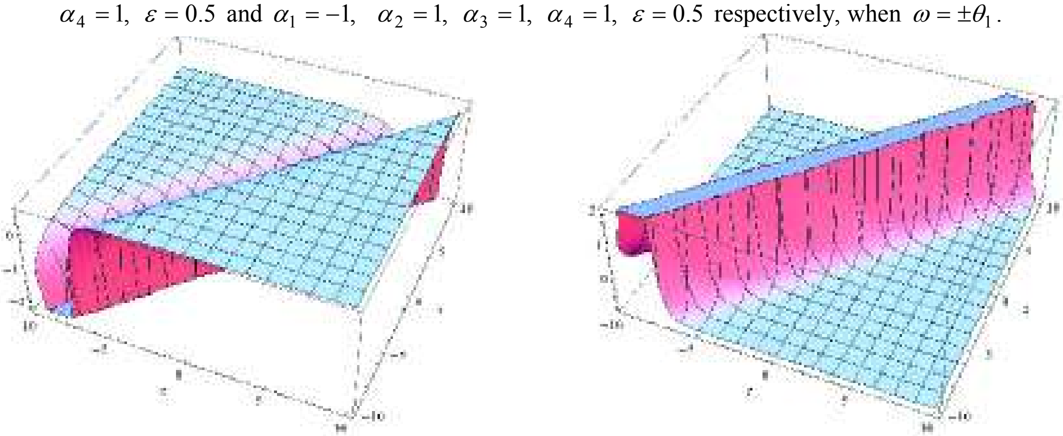

Fig. 17: Singular anti-bell shape and bell shape soliton u_{11} in (3.41) for $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ and $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$, $\varepsilon = 0.5$ respectively, when $\omega = \pm\theta_2$.

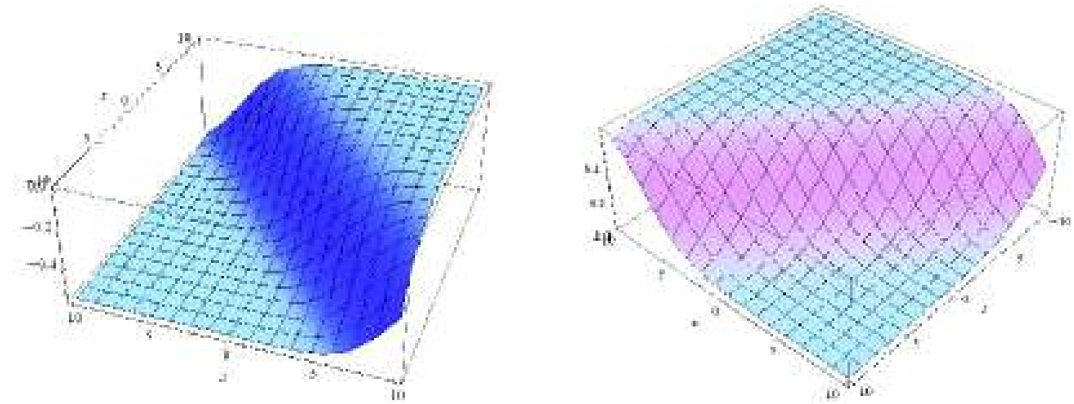

Fig. 18: Kink shape soliton u_{12} in (3.42) for $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 1,\ \alpha_4 = 1,\ \varepsilon = 0.5$ and $\alpha_1 = -1,\ \alpha_2 = 1,\ \alpha_3 = 1,\ \alpha_4 = 1,\ \varepsilon = 0.5$ respectively, when $\omega = \pm\theta_1$.

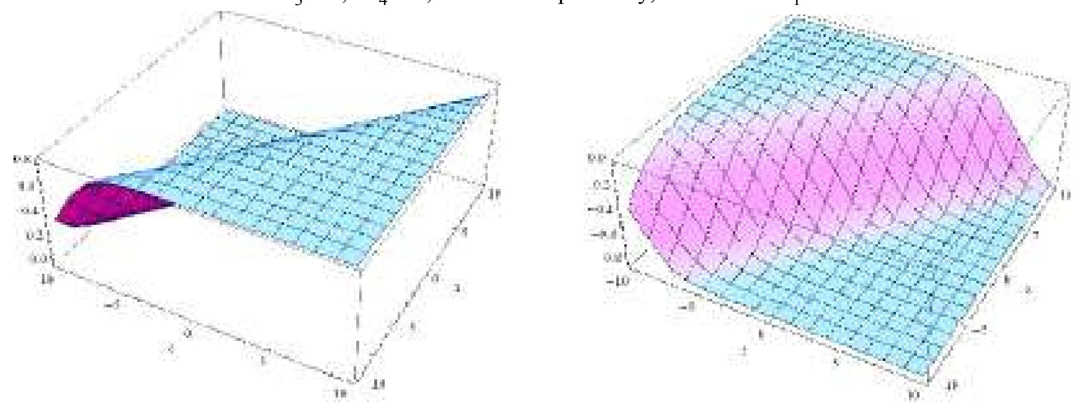

Fig. 19: Kink shape soliton u_{12} in (3.42) for $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 1,\ \alpha_4 = 1,\ \varepsilon = 0.5$ and $\alpha_1 = -1,\ \alpha_2 = 1,\ \alpha_3 = 1,\ \alpha_4 = 1,\ \varepsilon = 0.5$ respectively, when $\omega = \pm\theta_2$.

5. Conclusions

In this article, we considered the strain wave equation in microstructured solids whose balance number is two. If the balance number is greater than one, in general the MSE method does not provide any solution. For this case, we have established the procedure in order to implement the MSE method to solve NLEEs for balance number two. If the solution of $s(\xi)$ consists of polynomial of the wave variable ξ, it will not be the solitary wave solution, since it does not meet the condition $|u| \to 0$ as $\xi \to \pm\infty$ for solitary wave solution. In this case, each coefficient of the polynomial must be zero. This constraint is crucial to solve NLEEs for higher balance number. By using this achieved process, we solved the above mentioned NLEEs for non-dissipative case and dissipative case found some new traveling wave solutions. When the parameters receive special values, solitary wave solutions are derived from the exact solutions. We have analyzed the solitary wave properties of the solutions for different values of the physical parameters via the graphs.

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