Minireview Article

A Short Review of Crystallographic Point Group T: Orbits and Stabilizers for a Regular Tetrahedron

ABSTRACT

The clear definitions of symmetry operations (or elements) of the crystallographic point group T (tetrahedral group) were clearly pictured. For each element of the group, the permutation (P4) counterparts and the matrix representations were obtained. The orbits and stabilizers for vertices, edges, and faces of a regular tetrahedron under the action of T and, by this way, some subgroups of T were found.

Keywords: Crystallographic point group, Orbit, Tetrahedral symmetry, Stabilizer

1. INTRODUCTION

One of the most important application areas of the group theory is surely physics. In particular, in physics of crystals, finite symmetry groups provides an useful tool because it is known that the crystal classes, the 32 point groups, are constructed according to the symmetry groups which contain the operations of point symmetry such as rotation, reflection, rotation-reflection, and inversion. Group theory is closely related to symmetry. The symmetry is immunity to a possible change [1]. If thing under consideration is a crystal, then *the possible change* may be rotation about a symmetry axes, reflection from a mirror plane, or translation, etc., while *the immunity* is that the crystal under consideration remain unchanged under these symmetry operations.

Cayley's theorem states that every group is isomorphic to a group of permutations [2]. The isomorphism between a permutation group and a point group enables us to work on the symmetry groups of crystals. The effect of any symmetry element of point group of a crystal on the unit cell of the crystal is easily figured by means of its permutation counterpart in the permutation group which isomorphic to the point group of crystal under consideration.

2. SYMMETRY ELEMENTS IN TETRAHEDRAL POINT GROUP T

Crystallographic tetrahedral point group T is one of five point groups of cubic crystal system. This symmetry group contains 12 symmetry elements which leave a regular tetrahedron unchanged. To see these symmetry elements easily, let's put a regular tetrahedron into a cube whose edges are unit length as seen in Figure 1. The center of cube is fixed as the coordinate origin. The coordinates of vertices, which are labeled as 1, 2, 3, and 4, of regular tetrahedron are $\left({}^{1}/_{2},{}^{1}/_{2},{}^{1}/_{2}\right)$, $\left({}^{1}/_{2},{}^{-1}/_{2},{}^{-1}/_{2}\right)$, $\left({}^{-1}/_{2},{}^{1}/_{2},{}^{-1}/_{2}\right)$, and $\left({}^{-1}/_{2},{}^{-1}/_{2},{}^{1}/_{2}\right)$ respectively.

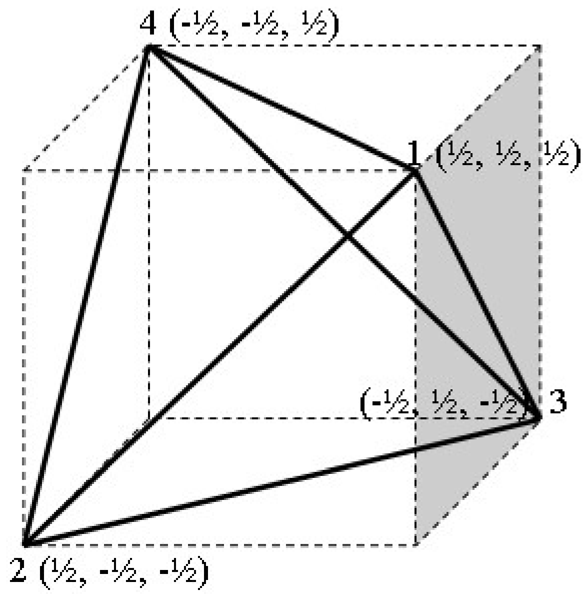

Fig. 1. A regular tetrahedron placed inside cube and its vertices together with their coordinates. The origin of coordinates is coincides with the center of cube and the lengths of edges of the cube are unit.

It is known from Cayley's theorem that the group T is isomorphic to a subgroup of the permutation group P(4) because P(4) has 4!=24 elements. We can assign a permutation counterpart for each element in T using this isomorphism. These permutation counterparts allow us to easily perform consecutive symmetry operations upon the regular tetrahedron. The first of 12 symmetry elements leaves the regular tetrahedron unchanged is identity element and its P(4) counterpart is $(1\ 2\ 3\ 4)$.

Three of the symmetry elements correspond to proper rotations by 180° about the axes which coincide with the coordinate axes (see Figure 2). Let us denominate them as C_{2x}, C_{2y}, and C_{2z}, where subscripts 2, *x*, *y*, and *z* stand for the proper rotations by 180° and proper rotations about *x*-, *y*-, and *z*-coordinate axes, respectively. Their permutation counterparts are $(2\ 1\ 4\ 3)$, $(3\ 4\ 1\ 2)$, and $(4\ 3\ 2\ 1)$, respectively. The remaining eight symmetry elements of group T consist of proper rotations by 120° about symmetry axes shown in Figure 3. They denoted by symbols C_{31}^{+}, C_{32}^{+}, C_{33}^{+}, C_{34}^{+}, C_{31}^{-}, C_{32}^{-}, C_{33}^{-}, and C_{34}^{-} where first subscript 3 corresponds to proper rotations by 120° and second subscripts 1, 2, 3, and 4 stands only for ranking of the axis of rotations while superscripts + and - denotes up and down axes of rotations. The P(4) counterparts of them are $(1\ 3\ 4\ 2)$, $(3\ 1\ 2\ 4)$, $(4\ 1\ 3\ 2)$, $(3\ 2\ 4\ 1)$, $(1\ 4\ 2\ 3)$, $(2\ 3\ 1\ 4)$, $(2\ 4\ 3\ 1)$, and $(4\ 2\ 1\ 3)$, respectively.

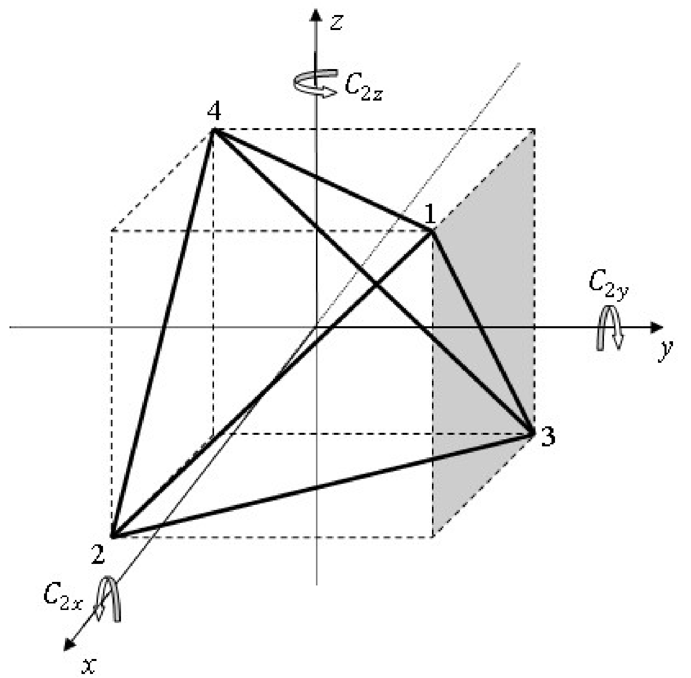

Fig. 2. Three symmetry operations leave a regular tetrahedron unchanged are the proper rotations by 180° about *x*-, *y*-, and *z*-axes. They are indicated with C_{2x}, C_{2y}, and C_{2z}.

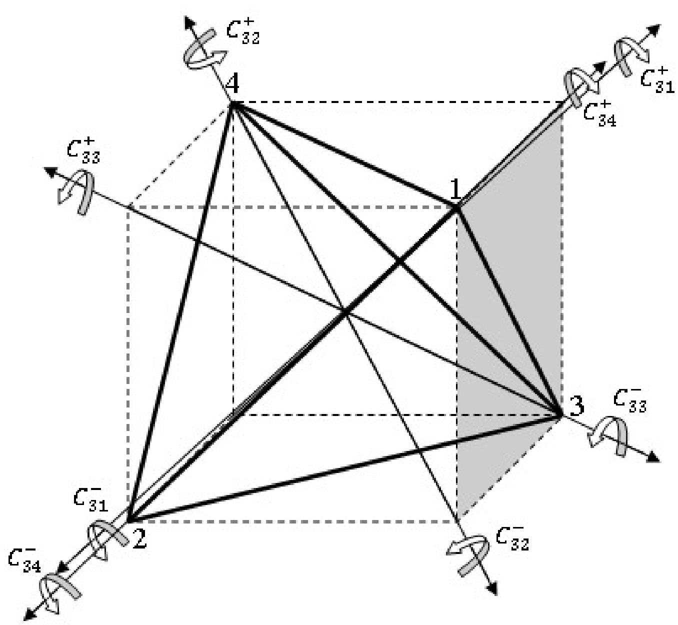

Fig. 3. Eight symmetry operations, C_{31}^+, C_{32}^+, C_{33}^+, C_{34}^+, C_{31}^-, C_{32}^-, C_{33}^-, and C_{34}^-, leave a regular tetrahedron unchanged under the proper rotations by 120°.

One can easily construct the Cayley table for group T using the permutation counterparts found above for each symmetry element in T. The multiplication of permutations means sequential implementations of the symmetry operations [3]. The Cayley table constructed using the permutation counterparts for each symmetry element in group T is given in Table 1.

Table 1. Cayley table for point group T.

	e	C_{2x}	C_{2y}	C_{2z}	C_{31}^+	C_{32}^+	C_{33}^+	C_{34}^+	C_{31}^-	C_{32}^-	C_{33}^-	C_{34}^-
e	e	C_{2x}	C_{2y}	C_{2z}	C_{31}^+	C_{32}^+	C_{33}^+	C_{34}^+	C_{31}^-	C_{32}^-	C_{33}^-	C_{34}^-
C_{2x}	C_{2x}	e	C_{2z}	C_{2y}	C_{33}^-	C_{34}^-	C_{34}^+	C_{33}^+	C_{32}^-	C_{31}^-	C_{31}^+	C_{32}^+
C_{2y}	C_{2y}	C_{2z}	e	C_{2x}	C_{32}^+	C_{31}^+	C_{32}^-	C_{31}^-	C_{34}^+	C_{33}^+	C_{34}^-	C_{33}^-
C_{2z}	C_{2z}	C_{2y}	C_{2x}	e	C_{34}^-	C_{33}^-	C_{31}^-	C_{32}^-	C_{33}^+	C_{34}^+	C_{32}^+	C_{31}^+
C_{31}^+	C_{31}^+	C_{32}^+	C_{34}^-	C_{33}^-	C_{31}^-	C_{33}^+	C_{2x}	C_{2z}	e	C_{2y}	C_{34}^+	C_{32}^-
C_{32}^+	C_{32}^+	C_{31}^+	C_{33}^-	C_{34}^-	C_{34}^+	C_{32}^-	C_{2z}	C_{2x}	C_{2y}	e	C_{31}^-	C_{33}^+
C_{33}^+	C_{33}^+	C_{31}^-	C_{34}^+	C_{32}^-	C_{2z}	C_{2y}	C_{33}^-	C_{32}^+	C_{34}^-	C_{31}^+	e	C_{2x}
C_{34}^+	C_{34}^+	C_{32}^-	C_{33}^+	C_{31}^-	C_{2y}	C_{2z}	C_{31}^+	C_{34}^-	C_{32}^+	C_{33}^-	C_{2x}	e
C_{31}^-	C_{31}^-	C_{33}^+	C_{32}^-	C_{34}^+	e	C_{2x}	C_{32}^+	C_{33}^-	C_{31}^+	C_{34}^-	C_{2z}	C_{2y}
C_{32}^-	C_{32}^-	C_{34}^+	C_{31}^-	C_{33}^+	C_{2x}	e	C_{34}^-	C_{31}^+	C_{33}^-	C_{32}^+	C_{2y}	C_{2z}
C_{33}^-	C_{33}^-	C_{34}^-	C_{32}^+	C_{31}^+	C_{32}^-	C_{34}^+	e	C_{2y}	C_{2x}	C_{2z}	C_{33}^+	C_{31}^-
C_{34}^-	C_{34}^-	C_{33}^-	C_{31}^+	C_{32}^+	C_{33}^+	C_{31}^-	C_{2y}	e	C_{2z}	C_{2x}	C_{32}^-	C_{34}^+

The matrix representation for each symmetry element in the crystallographic point group T can be easily found by means of the permutation counterparts of each element. The procedure of obtaining the matrix representations for any symmetry operation (or symmetry element in a symmetry group) is clearly described in [3]. The obtained matrix representations and permutation counterparts for all elements in group T is given Table 2.

Table 2. Symmetry elements in group T and their definitions, P(4) counterparts, and matrix representations.

Symmetry Element	Definition	P(4) Counterpart	Matrix representation	Symmetry Element	Definition	P(4) Counterpart	Matrix representation
e	Identity	(1234)	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	C_{33}^+	Proper rotation by 120° about C_{33}^+-axis (Fig. 3)	(4132)	$\begin{pmatrix}0&-1&0\\0&0&-1\\1&0&0\end{pmatrix}$
C_{2x}	Proper rotation by 180° about *x*-axis (Fig. 2)	(2143)	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$	C_{34}^+	Proper rotation by 120° about C_{34}^+-axis (Fig. 3)	(3241)	$\begin{pmatrix}0&-1&0\\0&0&1\\-1&0&0\end{pmatrix}$
C_{2y}	Proper rotation by 180° about *y*-axis (Fig. 2)	(3412)	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	C_{31}^-	Proper rotation by 120° about C_{31}^--axis (Fig. 3)	(1423)	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
C_{2z}	Proper rotation by 180° about *z*-axis (Fig. 2)	(4321)	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	C_{32}^-	Proper rotation by 120° about C_{32}^--axis (Fig. 3)	(2314)	$\begin{pmatrix}0&1&0\\0&0&-1\\-1&0&0\end{pmatrix}$
C_{31}^+	Proper rotation by 120° about C_{31}^+-axis (Fig. 3)	(1342)	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	C_{33}^-	Proper rotation by 120° about C_{33}^--axis (Fig. 3)	(2431)	$\begin{pmatrix}0&0&1\\-1&0&0\\0&-1&0\end{pmatrix}$
C_{32}^+	Proper rotation by 120° about C_{32}^+-axis (Fig. 3)	(3124)	$\begin{pmatrix}0&0&-1\\1&0&0\\0&-1&0\end{pmatrix}$	C_{34}^-	Proper rotation by 120° about C_{34}^--axis (Fig. 3)	(4213)	$\begin{pmatrix}0&0&-1\\-1&0&0\\0&1&0\end{pmatrix}$

3. ORBITS STABILIZERS AND SOME SUBGROUPS

According to the orbit-stabilizer theorem, if a group *G* acts on a set *X*, the set

$$Orb(x) = \{gx : g \in G\} \tag{1}$$

is called the *orbit* of *x*, such that $x \in X$ [2]. The set

$$Stab(x) = \{g \in G : gx = x\} \tag{2}$$

is called the *stabilizer* of element $x \in X$ and $Stab(x)$ is a subgroup of *G* [2].

Now, let's suppose that the point group

$$T = \left(e, C_{2x}, C_{2y}, C_{2z}, C_{31}^{+}, C_{32}^{+}, C_{33}^{+}, C_{34}^{+}, C_{31}^{-}, C_{32}^{-}, C_{33}^{-}, C_{34}^{-}\right)$$

acts on the set

$$X = \{V_1, V_2, V_3, V_4, E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}, F_{123}, F_{124}, F_{134}, F_{234}\}$$

where V_i stands for i*th* vertex, E_{ij} stands for the edge between vertices V_i and V_j, and F_{ijk} stands for the face constructed by vertices V_i, V_j and V_k in a regular tetrahedron. We immediately get, therefore, that

$$Orb(V_1) = Orb(V_2) = Orb(V_3) = Orb(V_4) = \{V_1, V_2, V_3, V_4\}$$

and

$$Stab(V_1) = \{e, C_{31}^{+}, C_{31}^{-}\}$$
$$Stab(V_2) = \{e, C_{34}^{+}, C_{34}^{-}\}$$
$$Stab(V_3) = \{e, C_{33}^{+}, C_{33}^{-}\}$$
$$Stab(V_4) = \{e, C_{32}^{+}, C_{32}^{-}\}$$

For the orbits and stabilizers of the edges of a regular tetrahedron, we can obtain that

$$\begin{aligned} Orb(E_{12}) &= Orb(E_{13}) = Orb(E_{14}) \\ &= Orb(E_{23}) = Orb(E_{24}) = Orb(E_{34}) \\ &= \{E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}\} \end{aligned}$$

and

$$Stab(E_{12}) = Stab(E_{34}) = \{e, C_{2x}\}$$
$$Stab(E_{13}) = Stab(E_{24}) = \left\{e, C_{2y}\right\}$$
$$Stab(E_{14}) = Stab(E_{23}) = \{e, C_{2z}\}$$

Finally, for the orbits and stabilizers of the faces of a regular tetrahedron, we get that

$$Orb(F_{123}) = Orb(F_{124}) = Orb(F_{134}) = Orb(F_{234}) = \{F_{123}, F_{124}, F_{134}, F_{234}\}$$

and

$$Stab(F_{123}) = \{e, C_{32}^{+}, C_{32}^{-}\}$$
$$Stab(F_{124}) = \{e, C_{33}^{+}, C_{33}^{-}\}$$
$$Stab(F_{134}) = \{e, C_{34}^{+}, C_{34}^{-}\}$$
$$Stab(F_{234}) = \{e, C_{31}^{+}, C_{31}^{-}\}$$

It is well known that the order of element $g \in G$, $|g|$, is n if following condition is provided.

$$g^n = e \qquad (3)$$

It can be found the orders of elements in group T using this condition as follows:

$$|e| = 1$$
$$|C_{2x}| = \left|C_{2y}\right| = |C_{2z}| = 2$$
$$|C_{31}^{+}| = |C_{32}^{+}| = |C_{33}^{+}| = |C_{34}^{+}| = |C_{31}^{-}| = |C_{32}^{-}| = |C_{33}^{-}| = |C_{34}^{-}| = 3$$

The period of element $g \in G$, $Per(g)$, forms set

$$Per(g) = \{e, g, g^2, \dots, g^{n-1}\} \qquad (4)$$

where n is the order of g and $Per(g)$ is an Abelian subgroups of G. The periods of elements in group T can be found using expression (4) as follows:

$$Per(e) = \{e\}$$

$$Per(C_{2x}) = \{e, C_{2x}\}$$

$$Per(C_{2y}) = \{e, C_{2y}\}$$

$$Per(C_{2z}) = \{e, C_{2z}\}$$

$$Per(C_{31}^{+}) = Per(C_{31}^{-}) = \{e, C_{31}^{+}, C_{31}^{-}\}$$

$$Per(C_{32}^{+}) = Per(C_{32}^{-}) = \{e, C_{32}^{+}, C_{32}^{-}\}$$

$$Per(C_{33}^{+}) = Per(C_{33}^{-}) = \{e, C_{33}^{+}, C_{33}^{-}\}$$

$$Per(C_{34}^{+}) = Per(C_{34}^{-}) = \{e, C_{34}^{+}, C_{34}^{-}\}$$

As seen from results found above, the stabilizers of vertices and faces of a regular tetrahedron are the periods of elements $C_{31}^{+}, C_{32}^{+}, C_{33}^{+}, C_{34}^{+}, C_{31}^{-}, C_{32}^{-}, C_{33}^{-}$, and C_{34}^{-} while the stabilizers of edges of a regular tetrahedron are same the periods of elements C_{2x}, C_{2y}, C_{2z}.

4. CONCLUSION

All symmetry elements in the crystallographic point group T and their actions on a regular tetrahedron are clearly pictured. P(4) counterparts and matrix representations of the symmetry elements in T are found and Cayley table for this point group is constructed. Also the periods of all symmetry elements in T are determined using the orders of elements. For each element in the set *X*, which consists of vertices, edges, and faces of a regular tetrahedron, the orbits and stabilizers are obtained under action of T. It is shown that the stabilizers equal to the periods of elements in T, which are the Abelian subgroups of T.

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