Bianchi Type-IX Cosmological Model ~~With~~ with a Perfect Fluid in the *f(R)* Theory of Gravity

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ABSTRACT

Bianchi type-IX space-time is considered in the framework of the $f(R)$ theory of gravity when the source ~~for~~ of the energy momentum tensor is a perfect fluid. The cosmological model is obtained by using the condition that the expansion scalar (θ) is proportional to the shear scalar (σ).The physical and geometrical properties of the model are also discussed.

Keywords: $f(R)$ gravity, Bianchi Type-IX space-time

1. INTRODUCTION

Cosmological observations in the late 1990's from different sources such as the Cosmic Microwave Background Radiation~~s~~ (CMBR) and ~~Supernovae~~ supernova (SN Ia) surveys indicate that the universe consist of 4% ordinary matter, 20% dark matter (DM) and 76% dark energy (DE) [1-4]. The DE has large negative pressure while the pressure of DM is negligible. Wald [5] has distinguished DM and DE and clarified that ordinary matter and DM satisfy the strong energy condition~~s~~, whereas DE does not. The DE resembles with a cosmological constant and the self-interaction potential of scalar fields. The scalar field is provided by the dynamically changing DE, including quintessence, k-essence, tachyon, phantom, ghost condensate and quintom etc. The study of high redshift supernova experiments [6-8], CMBR [9-10], ~~Large~~ large scale structure [11] and recent evidence~~s~~ from observational data [12-14] suggest that the universe is not only expanding but also accelerating.

Comment [A1]: These references are generic/historical. What recent evidence?

There are two major approaches ~~according~~ to the problem of accelerating expansion. One is to introduce a DE component in the universe and study its effects. ~~Other~~ The alternative is to modify general relativity; this is termed as modified gravity approach. We are interested in second ~~one~~approach. After the introduction of General Relativity (GR) in 1915, questions related to its limitations were in discussion. Einstein pointed out that Mach's principle is not substantiated by general relativity. Several attempts have been made to generalize the general theory of gravitation by incorporating Mach's principle and other desired features which were lacking in the original theory. Alternatives to ~~theories~~ Einstein's theory of gravitation have been proposed ~~to Einstein's theory to~~ , ~~incorporate~~ incorporating certain desirable features in the general theory. In ~~the last~~recent decades, as an alternative to general relativity, scalar-~~-~~tensor theories and modified theories of gravitation have been proposed. The most popular amongst them ~~are~~ include the theories of Brans-Dicke [15], Nordtvedt [16], Sen [17], Sen and Dunn [18], Wagonar [19], Saez-Ballester [20] etc. Recently, $f(R)$ gravity and $f(R,T)$ gravity theories have ~~much~~ gained importance among~~st~~ the modified theories of gravity because these theories are supposed to provide natural gravitational alternatives to dark energy. Among~~st~~ the various modifications, the $f(R)$ theory of gravity is treated most suitable due to cosmologically important $f(R)$ models. In $f(R)$ gravity, the Lagrangian density f is an arbitrary function of R [15, 21-23]. The model with $f(R)$ gravity can ~~laid~~ lead to the accelerated expansion of the universe. A generalization of $f(R)$ modified theory of gravity was proposed by Takahashi and Soda [24] by including explicit coupling of an arbitrary function of the Ricci ~~Scalar~~ scalar R with the matter Lagrangian density L_m. There are two formalisms ~~in~~ to deriving field equations from the action in $f(R)$ gravity. The first is the standard metric formalism in which the field equations are derived by the variation of the action with respect to the metric tensor $g_{\mu\nu}$. The second is the Palatini formalism. Maeda [25] have investigated Palatini formulation of the non-minimal geometry-coupling models. Multamaki and Vilja [26] obtained spherically symmetric solutions of modified field equations in $f(R)$ theory of gravity. Akbar and Cai [27] studied $f(R)$ theory of gravity action ~~is~~ as a nonlinear function of the curvature scalar R. Nojiri and Odinstove [28-30] derived the result that a unification of the early time inflation and late time acceleration is allowed in $f(R)$ theory. Ananda, Carloni and Dunsby [31] studied structure growth in $f(R)$ theory with a dust equation of state. Sharif and Shamir [32] and Sharif [33] have studied the vacuum solutions of Bianchi type-I, V and VI space-times. Sharif and Shamir [34] and Sharif and Kausar [35] obtained the non-vacuum solutions of Bianchi type-I, III and V space-times in $f(R)$ theory of gravity. Adhav [36, 37] have investigated the Kantowski-Sachs string cosmological model and the Bianchi type-III cosmological model with a perfect fluid in $f(R)$ gravity. Singh and Singh [38] have obtained a functional form of $f(R)$ with power-law expansion in Bianchi type-I space-times. Recently Jawad and Chattopadhyay [39] have investigated new holographic dark energy in $f(R)$ Horava-Lifshitz gravity. Rahman et al. [40] have obtained non-commutative wormholes in $f(R)$ gravity with Lorentzian distribution.

Motivated by the above investigations, in this paper an attempt is made to study Bianchi type-IX space-time when the universe is filled with a perfect fluid in the $f(R)$ theory of gravity with standard metric formalism. Bianchi type-IX space-time ~~are~~ is of vital importance in describing cosmological models ~~at~~ during the early stages of evolution of the universe~~.~~ .

This work is organized as follows: In Section 2, the $f(R)$ gravity formalism is ~~presented~~introduced. In Section 3, the model and field equations ~~have been~~are presented. The field equations ~~have been~~are solved in Section 4. The physical and geometrical behavior~~s~~ of the two models ~~have been~~are discussed in Section 5. ~~In~~ Section 6, contains concluding remarks ~~have been expressed~~.

2. $f(R)$ GRAVITY FORMALISM:

The action of $f(R)$ gravity is given by

$$S=\int\sqrt{-g}\left(\frac{1}{16\pi G}f(R)+L_m\right)d^4x\,. \qquad (1)$$

Here $f(R)$ is a general function of the Ricci scalar R and L_m is the matter Lagrangian. The corresponding field equations of ~~the~~ $f(R)$ gravity are found by varying the action with respect to the metric $g_{\mu\nu}$:

$$F(R)R_{\mu\nu}-\frac{1}{2}f(R)g_{\mu\nu}-\nabla_\mu\nabla_\nu F(R)+g_{\mu\nu}\Box F(R)=kT_{\mu\nu}, \qquad (2)$$

where $F(R)=\frac{d}{dR}f(R)$, $\Box\equiv\nabla^\mu\nabla_\nu$, ∇_μ is the covariant derivative and $T_{\mu\nu}$ is the standard matter energy-momentum tensor derived from the Lagrangian L_m.

Taking the trace of the above equation (with $k=1$), we obtain

$$F(R)R - 2f(R) + 3\,\Box\, F(R) = T\,. \tag{3}$$

On simplification, equation (3) leads to

$$f(R) = \frac{F(R)R + 3\nabla^{\mu}\nabla_{\mu}F - T}{2}\,. \tag{4}$$

3. METRIC AND FIELD EQUATIONS:

Bianchi type-IX metric is considered in the form,

$$ds^2 = -dt^2 + a^2dx^2 + b^2dy^2 + \left(b^2\sin^2 y + a^2\cos^2 y\right)dz^2 - 2a^2\cos y\,dx\,dz, \tag{5}$$

where a, b are scale factors and are functions of cosmic time t.

The Ricci scalar for Bianchi type-IX model is given by

$$R = -2\left[\frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} + 2\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{1}{b^2} - \frac{a^2}{4b^4}\right]. \tag{6}$$

The energy momentum tensor for the perfect fluid is given by

$$T_{ij} = (\rho + p)u_i u_j - pg_{ij}\,, \tag{7}$$

satisfying the barotropic equation of state

Comment [A2]: Two questions. First, is γ constant? (That is, if the perfect fluid is, say, an ideal gas, is it assumed to have constant temperature?) Second, is there any reason to include the case γ > 1/3?

$$p = \gamma\rho,\ 0 \le \gamma \le 1\,, \tag{8}$$

where ρ is the energy density and p is the pressure of the fluid.

In co-moving coordinate ~~systems~~s,

$$T_1^1 = T_2^2 = T_3^3 = -p,\ .\,T_4^4 = \rho,\quad T = \rho - 3p\,. \tag{9}$$

With the help of equations (7) to (9), the field equations (2) for the metric (5) are found:

Comment [A3]: Note that I reinserted (R) into expressions like F(R). Either drop it everywhere or use it everywhere, don't use inconsistent notation.

$$\left(\frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b}\right)F(R) + \frac{1}{2}f(R) + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)\dot{F}(R) = -\rho\,, \tag{10}$$

$$\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{a^2}{2b^4}\right)F(R) + \frac{1}{2}f(R) + \ddot{F}(R) + 2\frac{\dot{b}}{b}\dot{F}(R) = p\,, \tag{11}$$

$$\left(\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{1}{b^2} - \frac{a^2}{2b^4}\right)F(R) + \frac{1}{2}f(R) + \ddot{F}(R) + \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right)\dot{F}(R) = p\,, \tag{12}$$

where the overdot $(\dot{\ })$ denotes the differentiation with respect to t.

4. SOLUTIONS OF FIELD EQUATIONS:

The field equations (10) to (12) are highly non-linear differential equations in five unknowns, a, b, p, ρ, F. Hence, to obtain a well-~~determinate~~ determined solution of the system, we ~~take~~ assume that the square of the expansion scalar (θ) is proportional to the shear scalar (σ^2) ~~(Collins~~ ~~et al.~~ [41]~~)~~, which leads to

$$a = b^m, \ (m \neq 1), \tag{13}$$

where m is proportionality constant.

Also the power law relation between the scale factor (A) and scalar field (F) [37, 42-43] has been given by

$$F \, \alpha \, A^n, \tag{14}$$

where n is an arbitrary constant and A is the average scale factor.

Equation (14) leads to

$$F = K A^n, \tag{15}$$

where K is a proportionality constant.

With the help of equation (13), equation (15) reduces to

$$F = K b^{\frac{(m+2)n}{3}}. \tag{16}$$

~~Subtraction~~ Subtracting ~~of~~ equation (10) from (11)~~, (10) from~~ and (12), respectively and dividing the result by F gives

$$2\frac{\dot{a}}{a}\frac{\dot{b}}{b} - 2\frac{\ddot{b}}{b} + \frac{a^2}{2b^4} + \frac{\ddot{F}}{F} - \frac{\dot{a}}{a}\frac{\dot{F}}{F} = \frac{p+\rho}{F}, \tag{17}$$

$$\frac{\dot{a}}{a}\frac{\dot{b}}{b} - \frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{1}{b^2} - \frac{a^2}{2b^4} + \frac{\ddot{F}}{F} - \frac{\dot{b}}{b}\frac{\dot{F}}{F} = \frac{p+\rho}{F}. \tag{18}$$

~~Subtraction~~ Subtracting ~~of~~ equation (18) from equation (17) yields

$$\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} - \frac{\dot{a}}{a}\frac{\dot{F}}{F} + \frac{\dot{b}}{b}\frac{\dot{F}}{F} - \frac{1}{b^2} - \frac{\dot{b}^2}{b^2} + \frac{a^2}{b^4} = 0. \tag{19}$$

With the help of equations (13) and (16), equation (19) leads to

$$\frac{\ddot{b}}{b} + \frac{(3m^2 - m^2 n - mn + 2n - 3)}{3(m-1)}\frac{\dot{b}^2}{b^2} - \frac{1}{(m-1)b^2} + \frac{b^{2m-4}}{(m-1)} = 0. \tag{20}$$

On simplification, equation (20) reduces to

Comment [A4]: Can you explain why? What is the relationship between θ and σ on the one hand, and *a*, *b*, and *m* on the other?

Comment [A5]: This is the first time you use *A*. What is its relationship with the metric and its parameters? Show it.

Comment [A6]: I thought *F* was the first derivative of *f*(*R*)?

Comment [A7]: Average in what sense?

Comment [A8]: How do we know? You have not told us how *A* relates to *a* and *b*.

Comment [A9]: The first term in Eq. (21) should be *d*/*dt*, not *d*/*db*!

$$\frac{d}{db}(\dot{b}^2)+\frac{2(3m^2-m^2n-mn+2n-3)}{3(m-1)b}(\dot{b}^2)=\frac{1}{(m-1)}\left(2b^{-2}-2b^{2m-3}\right). \tag{21}$$

Integrating equation (21) yields

> **Comment [A10]:** How exactly did you do that? Eq. (21) is a second-order nonlinear ODE. It is not readily integrable.

$$\dot{b}=\sqrt{\frac{3}{(3m^2-m^2n-mn+2n-3)}-\frac{3}{(6m^2-m^2n-mn-6m+2n)}b^{2(m-1)}}\ . \tag{22}$$

Using equations (13) and (22), equation (5) reduces to

$$ds^2=\left\{\begin{array}{l} -\dfrac{1}{\left\{\dfrac{3}{(3m^2-m^2n-mn+2n-3)}-\dfrac{3}{(6m^2-m^2n-mn-6m+2n)}b^{2(m-1)}\right\}}db^2 \\ +b^{2m}dx^2+b^2dy^2+\left(b^2\sin^2 y+b^{2m}\cos^2 y\right)dz^2-2b^{2m}\cos y\,dx\,dz \end{array}\right\}. \tag{23}$$

Using ~~transformations~~ the new coordinates $b=T, x=X, y=Y, z=Z$, equation (23) leads to

> **Comment [A11]:** Be sure to replace *b* with *T* everywhere. (I edited Eq. (24)). Also, what is the reason for introducing *X*, *Y*, *Z*? What's wrong with keeping *x*, *y*, *z* and just replace *b* with *T*?

$$ds^2=\left\{\begin{array}{l} -\dfrac{1}{\left\{\dfrac{3}{(3m^2-m^2n-mn+2n-3)}-\dfrac{3}{(6m^2-m^2n-mn-6m+2n)}T^{2(m-1)}\right\}}dT^2 \\ +T^{2m}dX^2+T^2dY^2+\left(T^2\sin^2 Y+T^{2m}\cos^2 Y\right)dZ^2-2T^{2m}\cos Y\,dX\,dZ \end{array}\right\}. \tag{24}$$

5. SOME PHYSICAL PROPERTIES OF THE MODEL:

For the cosmological model (24), ~~the~~ physical quantities such as the spatial volume V, Hubble parameter H, expansion scalar θ, mean anisotropy parameter A_m, shear scalar σ^2, energy density ρ are obtained as follows.~~:~~

The ~~Spatial~~ spatial volume is in the form~~,~~

$$V = T^{m+2}. \tag{25}$$

The Hubble parameter is given by~~,~~

$$H = \frac{(m+2)}{3T}\left(\frac{3}{(3m^2 - m^2 n - mn + 2n - 3)} - \frac{3}{(6m^2 - m^2 n - mn - 6m + 2n)} T^{2(m-1)}\right)^{1/2}. \tag{26}$$

~~Expansion~~ The expansion scalar is~~,~~

$$\theta = \frac{(m+2)}{T}\left(\frac{3}{(3m^2 - m^2 n - mn + 2n - 3)} - \frac{3}{(6m^2 - m^2 n - mn - 6m + 2n)} T^{2(m-1)}\right)^{1/2}. \tag{27}$$

The ~~Mean~~ mean ~~Anisotropy~~ anisotropy ~~Parameter~~parameter~~,~~ is

$$A_m = \frac{2(m-1)^2}{(m+2)^2} = \text{constant} \ (\neq 0 \text{ for } m \neq 1). \tag{28}$$

~~Shear~~ The shear scalar is given by~~,~~

$$\sigma^2 = \left\{\frac{(m-1)^2}{(3m^2 - m^2 n - mn + 2n - 3)} T^{-2} - \frac{(m-1)^2}{(6m^2 - m^2 n - mn - 6m + 2n)} T^{2(m-2)}\right\}. \tag{29}$$

We observe that

$$\frac{\sigma^2}{\theta^2} = \frac{(m-1)^2}{3(m+2)^2} = \text{constant}(\neq 0), \text{ for } m \neq 1. \tag{30}$$

Using equations (8), (16) and (22) in equation (10), the energy density is obtained as

$$\rho = \frac{K}{2(1+\gamma)} T^{\frac{mn+2n-6}{3}} \left(\begin{array}{l} 1 + \dfrac{9(1+4m-m^2) + (mn+2n)(2mn-3m+4n-9)}{3(3m^2 - m^2 n - mn + 2n - 3)} \\ - \dfrac{\left[\begin{array}{l} 9(1+4m-m^2) \\ + (mn+2n-3m+3)(2mn-3m+4n-9) \end{array}\right]}{3(6m^2 - m^2 n - mn - 6m + 2n)} T^{2(m-1)} \end{array}\right). \tag{31}$$

From equation (6) we obtain

$$R = \left(\begin{array}{l} \left[1 + \dfrac{3(1+m+m^2)}{(3m^2 - m^2 n - mn + 2n - 3)}\right] - \left[\dfrac{3(1+m+m^2)}{(6m^2 - m^2 n - mn - 6m + 2n)} + \dfrac{2(m+3)}{2(m-1)} \\ \dfrac{2(m+2)(3m^2 - m^2 n - mn + 2n - 3)}{(m-1)(6m^2 - m^2 n - mn - 6m + 2n)}\right] T^{2(m-2)} \end{array}\right). \tag{32}$$

~~From e~~Equation (4) leads to the following expression for the function ~~of Ricci scalar~~ $f(R)$ ~~leads to~~of the Ricci scalar:

$$f(R)=\left\{\begin{array}{l}\left[\begin{array}{l}1+\dfrac{3(1+4m-m^2)+(mn+2n)(mn+2n-3)}{(3m^2-m^2n-mn+2n-3)}-\dfrac{(1-3\gamma)}{2(1+\gamma)}\\ \left(1+\dfrac{9(1+4m-m^2)+(mn+2n)(2mn-3m+4n-9)}{3(3m^2-m^2n-mn+2n-3)}\right)\end{array}\right]\dfrac{K}{2}T^{\frac{(m+2)n-6}{3}}-\\ \left[\begin{array}{l}\dfrac{3(1+m+m^2)+(mn+2n)(mn+2n-3)}{(6m^2-m^2n-mn-6m+2n)}+\dfrac{(3m+9-2mn-4n)}{2(m-1)}\\ +\dfrac{(mn+2m+2n+4)(3m^2-m^2n-mn+2n-3)}{(m-1)(6m^2-m^2n-mn-6m+2n)}\\ -\dfrac{(1-3\gamma)}{2(1+\gamma)}\left(\dfrac{9(1+4m-m^2)+(mn+2n-3m+3)(2mn-3m+4n-9)}{3(6m^2-m^2n-mn-6m+2n}\right)\end{array}\right]\dfrac{K}{2}T^{\frac{(m+2)n+6(m-2)}{3}}\end{array}\right\}$$

, (33)

which clearly indicates that $f(R)$ is ~~written in terms of T, which is true as $f(R)$~~ depends upon T only.

~~By inserting the value of R from equation (32) in equation (33), $f(R)$ reduces to a function of R.~~

~~For a~~In the special case when $m=n=2$, $f(R)$ turns out to be

$$f(R)=\frac{K}{3(1+\gamma)}\left(\frac{88}{59+4R}\right)\left[-41+15\gamma+\frac{220(2-3\gamma)}{59+4R}\right]. \quad (34)$$

This gives $f(R)$ only as a function of R. This gives $f(R)$ explicitly as a function of R only.

Comment [A12]: There is no need to show something that is true by definition. You just managed to eliminate R from the expression, which is good.

Comment [A13]: Of course $f(R)$ is a funciton of R only, you defined it that way. What's the point?

6. CONCLUSION:

A Bianchi type-IX cosmological model have been obtained when universe is filled with a perfect fluid in $f(R)$ theory of gravity. The ~~model~~ obtained model ~~has singularity~~is singular at $T=0$ and the physical parameters H, θ, and σ^2 are ~~infinite~~ divergent at $T=0$ as well. ~~It is~~We observed that the scale factors and volume of the model vanishe~~s~~ at the initial epoch and increase~~s~~ with the passage of time representing an expanding universe. From equation (26) and (28) the mean anisotropy parameter A_m is shown to be constant and $\frac{\sigma^2}{\theta^2}(\neq 0)$ is also constant, hence the model is anisotropic throughout the evolution of the universe except ~~at~~ when $m=1$-; *i.e.* the model does not approach isotropy.

It is worth ~~to~~ mentioning that~~,~~ the obtained model ~~obtained~~ is point type singular, expanding, shearing, non-rotating and does not approach isotropy for large T. We hope that our model will be useful in the study of structure formation in the early universe and ~~an~~ the accelerating expansion of the universe at present.

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