# The Modified Simple Equation Method and Its Application to Solve NLEEs Associated with Engineering Problem

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## ABSTRACT

The modified simple equation (MSE) method is an important mathematical tool for searching closed-form solutions to nonlinear evolution equations (NLEEs). Earlier the method cannot be used to NLEEs for higher balance number. Very recently Khan and Akbar developed a technique to fulfill this shortcoming and solved NLEEs for balance number two by the MSE method. In the present paper, by using the MSE method, we derive some impressive solitary wave solutions to NLEES via the strain wave equation in microstructured solids which is a very important equation in the field of engineering. The solutions contain some free parameters and for particulars values of the parameters some known solutions are derived. The solutions exhibit necessity and reliability of the method.

**Keywords:** Modified simple equation method; balance number; solitary wave solutions; strain wave equation; microstructured solids.

**Mathematics Subject Classification**: 35C07, 35C08, 35P99.

## 1. INTRODUCTION

Physical systems are in general explained with nonlinear partial differential equations. The mathematical modeling of microstructured solid materials that change over time depends closely on the study of a variety of systems of ordinary and partial differential equations.

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Similar models are developed in diverse fields of study, ranging from the natural and physical sciences, population ecology to economics, infectious disease epidemiology, neural networks, biology, mechanics etc. In spite of the eclectic nature of the fields wherein these models are formulated, different groups of them contribute adequate common attributes that make it possible to examine them within a unified theoretical structure. Such study is an area of functional analysis, usually called the theory of evolution equations. Therefore, the investigation of solutions to NLEEs plays a very important role to uncover the obscurity of many phenomena and processes throughout the natural sciences. However, one of the essential problems is to obtain theirs closed-form solutions. For that reason, diverse groups of engineers, physicists, and mathematicians have been working tirelessly to investigate closed-form solutions to NLEEs. Accordingly, in the recent years, they establish several methods to search exact solutions, for instance, the Darboux transformation method [1], the Jacobi elliptic function method [2, 3], the He's homotopy perturbation method [4, 5], the tanh-function method [6, 7], the extended tanh-function method [8, 9], the Lie group symmetry method [10], the variational iteration method [11], the Hirota's bilinear method [12], the Backlund transformation method [13, 14], the inverse scattering transformation method [15], the sine-cosine method [16, 17], the Painleve expansion method [18], the Adomian decomposition method [19, 20], the $(G'/G)$-expansion method [21-26], the first integration method [27], the F-expansion method [28], the auxiliary equation method [29], the ansatz method [30, 31], the Exp-function method [32, 33], the homogeneous balance method [34], the modified simple equation method [35-39], the $\exp(-\varphi(\eta))$-expansion method [40, 41], the Miura transformation method [42], and others.

Microstructured materials like crystallites, alloys, ceramics, and functionally graded materials have gained broad application. The modeling of wave propagation in such materials should be able to account for various scales of microstructure [43]. In the past years, many authors have studied the strain wave equation in microstructured solids, such as, Alam et al. [43] solved this equation by using the new generalized $(G'/G)$-expansion method. Pastrone et

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al. [44], Porubov and Pastrone [45] examined bell-shaped and kink-shaped solutions of this engineering problem. Akbar et al. [46] constructed traveling wave solutions of this equation by using the generalized and improved $(G'/G)$-expansion method. The above analysis shows that several methods to achieve exact solutions to this equation have been accomplished in the recent years. But the equation has not been studied by means of the MSE method. In this article, our aim is, we will apply the MSE method following the technique derived in the Ref. [47] to examine some new and impressive solitary wave solutions to the strain wave equation in microstructured solids.

The structure of this article is as follows: In section 2, we describe the method. In section 3, we apply the MSE method to the strain wave equation in microstructured solids. In section 4, we provide the physical interpretations of the obtained solutions. Finally, in section 5, conclusions are given.

## 2. DESCRIPTION OF THE METHOD

Assume the nonlinear evolution equation has the following form

$$P(u, u_t, u_x, u_y, u_z, u_{xx}, u_{tt}, ...) = 0, \qquad (2.1)$$

where $u = u(x, y, z, t)$ is an unidentified function, $P$ is a polynomial function in $u = u(x, y, z, t)$ and its partial derivatives, wherein nonlinear term of the highest order and the highest order linear terms exist and subscripts indicate partial derivatives. To solve (2.1) by using the MSE method [35-39], we need to perform the subsequent steps:

**Step 1**: Now, we combine the real variable $x$ and $t$ by a compound variable $\xi$ as follows:

$$u(x, y, z, t) = U(\xi), \qquad \xi = x + y + z \pm \omega\, t. \qquad (2.2)$$

Here $\xi$ is called the wave variable it allows us to switch Eq. (2.1) into an ordinary differential equation (ODE):

$$Q(U, U', U'', U''', \cdots), \qquad (2.3)$$

where $Q$ is a polynomial in $U(\xi)$ and its derivatives, where $U'(\xi) = \frac{dU}{d\xi}$.

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**Step 2**: We assume that Eq. (2.3) has the traveling wave solution in the following form,

$$U(\xi) = \sum_{i=0}^{N} a_i \left( \frac{\psi'(\xi)}{\psi(\xi)} \right)^i, \tag{2.4}$$

where $a_i$, $(i = 0,1,2,\cdots,N)$ are arbitrary constants, such that $a_N \neq 0$, and $\psi(\xi)$ is an unidentified function which is to be determined later. In $(G'/G)$-expansion method, Exp-function method, tanh-function method, sine-cosine method, Jacobi elliptic function method etc., the solutions are initiated through several auxiliary functions which are previously known, but in the MSE method, $\psi(\xi)$ is neither a pre-defined function nor a solution of any pre-defined differential equation. Therefore, it is not possible to speculate from formerly, what kind of solution can be found by this method.

**Step 3**: We determine the positive integer $N$, come out in Eq. (2.4) by taking into account the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order occurring in Eq. (2.3).

**Step 4**: We calculate the necessary derivatives $U', U'', U'''$, etc., then insert them into Eq. (2.3) and then taken into consideration the function $\psi(\xi)$. As a result of this insertion, we obtain a polynomial in $(\psi'(\xi)/\psi(\xi))$. We equate all the coefficients of $(\psi(\xi))^{i}$, $(i = 0,1,2,...,N)$ to this polynomial to zero. This procedure yields a system of algebraic and differential equations whichever can be solved for getting $a_i$ $(i = 0,1,2,\cdots,N)$, $\psi(\xi)$ and the value of the other parameters.

## 3. APPLICATION OF THE METHOD

In this section, we will execute the application of the MSE method to extract solitary wave solutions to the strain wave equation in microstructured solids which is a very important equation in the field of engineering. Let us consider the strain wave equation in microstructured solids:

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$$u_{tt} - u_{xx} - \varepsilon\, \alpha_1 (u^2)_{xx} - \gamma \alpha_2 u_{xxt} + \delta \alpha_3 u_{xxxx} - (\delta \alpha_4 - \gamma^2 \alpha_7) u_{xxtt} + \gamma \delta (\alpha_5 u_{xxxxt} + \alpha_6 u_{xxtt}) = 0. \qquad (3.1)$$

## 3.1. THE NON-DISSIPATIVE CASE

The system is non-dissipative, if $\gamma = 0$ and determined by the double dispersive equation (see [44], [45], [48], [49] for details)

$$u_{tt} - u_{xx} - \varepsilon\, \alpha_1 (u^2)_{xx} + \delta \alpha_3 u_{xxxx} - \delta \alpha_4 u_{xxtt} = 0. \qquad (3.2)$$

The balance between dispersion and nonlinearities happen when $\delta = O(\varepsilon)$. Therefore, (3.2) becomes

$$u_{tt} - u_{xx} - \varepsilon\, \{\alpha_1 (u^2)_{xx} - \alpha_3 u_{xxxx} + \alpha_4 u_{xxtt}\} = 0. \qquad (3.3)$$

In order to extract solitary wave solutions of the strain wave equation in microstructured solids by using the MSE method, we use the traveling wave variable

$$u(x,t) = U(\xi), \quad \xi = x - \omega\, t. \qquad (3.4)$$

The wave transformation (3.4) reduces Eq. (3.3) into the ODE in the following form:

$$(\omega^2 - 1)\, U'' - \varepsilon \left\{ \alpha_1 (\, U^2)'' - (\alpha_3 - \omega^2 \alpha_4)\, U^{(iv)} \right\} = 0. \qquad (3.5)$$

where primes indicate differential coefficients with respect to $\xi$. Eq. (3.5) is integrable, therefore, integration (3.5) as many time as possible, we obtain the following ODE:

$$(\omega^2 - 1)\, U - \varepsilon \{\alpha_1 U^2 - (\alpha_3 - \omega^2 \alpha_4)\, U''\} = 0. \qquad (3.6)$$

where the integration constants are set zero, as we are seeking solitary wave solutions. Taking homogeneous balance between the terms $U''$ and $U^2$ appearing in Eq. (3.6), we obtain $N = 2$. Therefore, the shape of the solution of Eq. (3.6) becomes

$$U(\xi) = a_0 + \frac{a_1 \psi'}{\psi} + \frac{a_2 (\psi')^2}{\psi^2}. \qquad (3.7)$$

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wherein $a_0$, $a_1$ and $a_2$ are constants to be find out afterward such that $a_2 \neq 0$, and $\psi(\xi)$ is an unknown function. The derivatives of $U$ are given in the following:

$$U' = -\frac{a_1(\psi')^2}{\psi^2} - \frac{2a_2(\psi')^3}{\psi^3} + \frac{a_1\psi''}{\psi} + \frac{2a_2\psi'\psi''}{\psi^2}. \tag{3.8}$$

$$U'' = a_1\left\{\frac{2(\psi')^3}{\psi^3} - \frac{3\psi'\psi''}{\psi^2} + \frac{\psi'''}{\psi}\right\} + 2a_2\left\{\frac{(\psi'')^2}{\psi^2} + \frac{\psi'\psi'''}{\psi^2} - \frac{5(\psi')^2\psi''}{\psi^3} + \frac{3(\psi')^4}{\psi^4}\right\}. \tag{3.9}$$

Inserting the values of $U, U'$ and $U''$ into Eq. (3.6), and setting each coefficient of $\psi^j,\ j = 0, 1, 2, \cdots$ to zero, we derive, successively

$$a_0(-1 + \omega^2 - \varepsilon\, a_0\alpha_1) = 0. \tag{3.10}$$

$$a_1\{(-1 + \omega^2 - 2\varepsilon a_0\alpha_1)\psi' + \varepsilon(\alpha_3 - \omega^2\alpha_4)\psi'''\} = 0. \tag{3.11}$$

$$-\varepsilon a_1\psi'\{a_1\alpha_1\psi' + 3(\alpha_3 - \omega^2\alpha_4)\psi''\} + 2a_2\varepsilon(\alpha_3 - \omega^2\alpha_4)\psi'\psi'''$$
$$+ a_2\{(-1 + \omega^2 - 2\varepsilon a_0\alpha_1)(\psi')^2 + 2\varepsilon(\alpha_3 - \omega^2\alpha_4)(\psi'')^2\} = 0. \tag{3.12}$$

$$-2\varepsilon(\psi')^2\{a_1(a_2\alpha_1 - \alpha_3 + \omega^2\alpha_4)\psi' + 5a_2(\alpha_3 - \omega^2\alpha_4)\psi''\} = 0. \tag{3.13}$$

$$-\varepsilon a_2(a_2\alpha_1 - 6\alpha_3 + 6\omega^2\alpha_4)(\psi')^4 = 0. \tag{3.14}$$

From Eq. (3.10) and Eq. (3.14), we obtain

$$a_0 = 0,\quad \frac{-1 + \omega^2}{\varepsilon\alpha_1} \quad \text{and} \quad a_2 = \frac{6(\alpha_3 - \omega^2\alpha_4)}{\alpha_1}, \quad \text{scince } a_2 \neq 0.$$

Therefore, for the values of $a_0$, there arise the following cases:

**Case 1:** When $a_0 = 0$, from Eqs. (3.11)-(3.13), we obtain

$$a_1 = \pm\frac{6\sqrt{1 - \omega^2}\sqrt{\alpha_3 - \omega^2\alpha_4}}{\sqrt{\varepsilon}\alpha_1}$$

and

$$\psi(\xi) = c_2 + \frac{\varepsilon c_1(-\alpha_3 + \omega^2\alpha_4)}{-1 + \omega^2}\, e^{\mp\frac{\xi\sqrt{1-\omega^2}}{\sqrt{\varepsilon}\sqrt{\alpha_3 - \omega^2\alpha_4}}},$$

where $c_1$ and $c_2$ are integration constants.

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Substituting the values of $a_0, a_1, a_2$ and $\psi(\xi)$ into Eq. (3.7), we obtain the following exponential form solution:

$$U(\xi) = \frac{6e^{\pm\frac{\xi\sqrt{1-\omega^2}}{\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}}(-1+\omega^2)^2 c_1 c_2(-\alpha_3+\omega^2\alpha_4)}{\alpha_1\left((-1+\omega^2)c_2\, e^{\pm\frac{i\xi\sqrt{-1+\omega^2}}{\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}} + \varepsilon c_1(-\alpha_3+\omega^2\alpha_4)\right)^2}. \qquad (3.15)$$

Simplifying the required solution (3.15), we derive the following close-form solution of the strain wave equation in microstructured solids (3.3):

$$u(x,t) = \{6(-1+\omega^2)^2 c_1 c_2(-\alpha_3+\omega^2\alpha_4)\}$$

$$/\left[\alpha_1\left\{\pm i\sin\big((x-t\omega)\beta\big)\{(-1+\omega^2)c_2+\varepsilon c_1(\alpha_3-\omega^2\alpha_4)\}\right.\right.$$

$$\left.\left.+\cos\big((x-t\omega)\beta\big)\{(-1+\omega^2)c_2+\varepsilon c_1(-\alpha_3+\omega^2\alpha_4)\}\right\}^2\right] \qquad (3.16)$$

where $\beta \frac{\sqrt{-1+\omega^2}}{2\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}$. Solution (3.16) is the generalized solitary wave solution of the strain wave equation in microstructured solids. Since $c_1$ and $c_2$ are arbitrary constants, one might arbitrarily choose their values. Therefore, if we choose $c_1 = (-1+\omega^2)$ and $c_2 = \varepsilon(-\alpha_3 +$ $\omega 2\alpha 4$ then from (3.16), we obtain the following bell shaped soliton solution:

$$u_1(x,t) = \frac{3(-1+\omega^2)}{2\varepsilon\alpha_1}\operatorname{sech}^2\left(\frac{(x-t\omega)\sqrt{-1+\omega^2}}{2\sqrt{\varepsilon}\sqrt{-\alpha_3+\omega^2\alpha_4}}\right). \qquad (3.17)$$

Again, if we choose $c_1 = (-1+\omega^2)$ and $c_2 = -\varepsilon(-\alpha_3+\omega^2\alpha_4)$, then from (3.16), we obtain the following singular soliton:

$$u_2(x,t) = -\frac{3(-1+\omega^2)}{2\varepsilon\alpha_1}\operatorname{csch}^2\left(\frac{(x-t\omega)\sqrt{-1+\omega^2}}{2\sqrt{\varepsilon}\sqrt{-\alpha_3+\omega^2\alpha_4}}\right). \qquad (3.18)$$

On the other hand, when $c_1 = (-1+\omega^2)$ and $c_2 = \pm i\,\varepsilon(-\alpha_3+\omega^2\alpha_4)$, from solution (3.16), we obtain the following trigonometric solution:

$$u_3(x,t) = \frac{3(-1+\omega^2)}{2\varepsilon\alpha_1}\sec^2\left[\frac{1}{4}\left\{\pi + \frac{2(x-t\omega)\sqrt{-1+\omega^2}}{\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}\right\}\right]. \qquad (3.19)$$

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Again when $c_1 = (-1+\omega^2)$ and $c_2 = \mp i\,\varepsilon(-\alpha_3+\omega^2\alpha_4)$, then the generalized solitary wave solution (3.16) can be simplified as:

$$u_4(x,t) = \frac{3(-1+\omega^2)}{2\varepsilon\alpha_1}\csc^2\left[\frac{1}{4}\left\{\pi + \frac{2(-x+t\omega)\sqrt{-1+\omega^2}}{\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}\right\}\right]. \tag{3.20}$$

If we choose more different values of $c_1$ and $c_2$, we may derive a lot of general solitary wave solutions to the Eq. (3.3) through the MSE method. For succinctness, other solutions have been overlooked.

**Case 2:** When $a_0 = \dfrac{-1+\omega^2}{\varepsilon\alpha_1}$, then Eqs. (3.11)-(3.13) yield

$$a_1 = \pm\frac{6\sqrt{-1+\omega^2}\sqrt{\alpha_3-\omega^2\alpha_4}}{\sqrt{\varepsilon}\alpha_1}$$

And

$$\psi(\xi) = c_2 + \frac{\varepsilon c_1(\alpha_3-\omega^2\alpha_4)}{-1+\omega^2}\,e^{\mp\frac{\xi\sqrt{-1+\omega^2}}{\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}},$$

where $c_1$ and $c_2$ are constants of integration.

Now, by means of the values of $a_0$, $a_1$, $a_2$ and $\psi(\xi)$, from Eq. (3.7), we obtain the subsequent solution:

$$U(\xi) = \frac{-1+\omega^2}{\varepsilon\alpha_1} + \frac{6(-1+\omega^2)^2 c_1 c_2(-\alpha_3+\omega^2\alpha_4)e^{\pm\frac{\xi\sqrt{-1+\omega^2}}{\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}}}{\alpha_1\left\{(-1+\omega^2)c_2 e^{\pm\frac{\xi\sqrt{-1+\omega^2}}{\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}} + \varepsilon c_1(\alpha_3-\omega^2\alpha_4)\right\}^2}. \tag{3.21}$$

Now, transforming the required exponential function solution (3.21) into hyperbolic function, we obtain the following solution to the strain wave equation in the microstructured solids:

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$$u(x,t) = (-1+\omega^2)\big[(-1+\omega^2)^2\{\cosh(2\rho(x-t\omega)) + \sinh(2\rho(x-t\omega))\}c_2^2$$
$$+ \varepsilon^2\{\cosh(2\rho(x-t\omega)) - \sinh(2\rho(x-t\omega))\}c_1^2(\alpha_3-\omega^2\alpha_4)^2$$
$$+ 4\varepsilon(-1+\omega^2)c_1c_2(-\alpha_3+\omega^2\alpha_4)\big]$$
$$/\Big(\varepsilon\alpha_1\big[(-1+\omega^2)\{\cosh(\rho(x-t\omega)) + \sinh(\rho(x-t\omega))\}c_2$$
$$+ \varepsilon\{\cosh(\rho(x-t\omega)) - \sinh(\rho(x-t\omega))\}c_1(\alpha_3-\omega^2\alpha_4)\big]^2\Big). \qquad (3.22)$$

Thus, we acquire the generalized solitary wave solution (3.22) to the strain wave equation in microstructured solids, where $\rho = \frac{\sqrt{-1+\omega^2}}{2\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}$. Since $c_1$ and $c_2$ are integration constants, therefore, somebody might randomly pick their values. So, if we pick $c_1 = (-1+\omega^2)$ and $c_2 = -\varepsilon(\alpha_3-\omega^2\alpha_4)$, then from (3.22), we obtain the subsequent solitary wave solution:

$$u_5(x,t) = \frac{(-1+\omega^2)}{2\varepsilon\alpha_1}\left\{2 + 3\,\mathrm{csch}^2\left(\frac{(x-t\omega)\sqrt{-1+\omega^2}}{2\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}\right)\right\}. \qquad (3.23)$$

Again, if we pick $c_1 = (-1+\omega^2)$ and $c_2 = \varepsilon(\alpha_3-\omega^2\alpha_4)$, then the solitary wave solution (3.22) reduces to:

$$u_6(x,t) = -\frac{(-1+\omega^2)}{2\varepsilon\alpha_1}\left\{-2 + 3\,\mathrm{sech}^2\left(\frac{(x-t\omega)\sqrt{-1+\omega^2}}{2\sqrt{\varepsilon}\sqrt{\alpha_3-\omega^2\alpha_4}}\right)\right\}. \qquad (3.24)$$

Moreover, if we pick $c_1 = (-1+\omega^2)$ and $c_2 = \mp i\,\varepsilon(\alpha_3-\omega^2\alpha_4)$, then from (3.22), we derive the following solution:

$$u_7(x,t) = \frac{(-1+\omega^2)}{\varepsilon\alpha_1}\left\{1 - \frac{3}{2}\csc^2\left(\frac{\pi}{4} - \frac{1}{2}\frac{(x-t\omega)\sqrt{-1+\omega^2}}{\sqrt{\varepsilon}\sqrt{-\alpha_3+\omega^2\alpha_4}}\right)\right\}. \qquad (3.25)$$

Again, if we pick $c_1 = (-1+\omega^2)$ and $c_2 = \pm i\,\varepsilon(\alpha_3-\omega^2\alpha_4)$, then from (3.22), we obtain the following solution:

$$u_8(x,t) = \frac{(-1+\omega^2)}{\varepsilon\alpha_1}\left\{1 - \frac{3}{2}\csc^2\left(\frac{\pi}{4} + \frac{1}{2}\frac{(x-t\omega)\sqrt{-1+\omega^2}}{\sqrt{\varepsilon}\sqrt{-\alpha_3+\omega^2\alpha_4}}\right)\right\}. \qquad (3.26)$$

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Forasmuch as, $c_1$ and $c_2$ are arbitrary constants, if we choose more different values of them, we may derive a lot of general solitary wave solutions to the Eq. (3.3) through the MSE method easily. But, we did not write down the other solutions for minimalism.

**Remark 1**: Solutions (3.17)-(3.20) and (3.23)-(3.26) have been confirmed by inserting them into the main equation and found accurate.

### 3.2. THE DISSIPATIVE CASE

If $\gamma \neq 0$, then the system is dissipative. Therefore, for $\delta = \gamma = O(\varepsilon)$, the balance should be between nonlinearity, dispersion and dissipation, perturbed by the higher order dissipative terms to the strain wave equation in microstructured solids (see [44], [45], [48], [49] for details),

$$u_{tt} - u_{xx} - \varepsilon\{\alpha_1(u^2)_{xx} + \alpha_2 u_{xxt} - \alpha_3 u_{xxxx} + \alpha_4 u_{xxtt}\} = 0. \tag{3.27}$$

where $\varepsilon \to 0$, so the higher order term are omitted.

The traveling wave transformation (3.4) reduces Eq. (3.27) to the following ODE:

$$(\omega^2 - 1)\, U'' - \varepsilon\{\alpha_1(U^2)'' - \omega\, \alpha_2 U''' - (\alpha_3 - \omega^2\alpha_4)\, U^{(iv)}\} = 0. \tag{3.28}$$

where prime stands for the differential coefficient. Integrating Eq. (3.28) with respect to $\xi$, we get

$$(\omega^2 - 1)\, U - \varepsilon\{\alpha_1 U^2 - \omega\, \alpha_2 U' - (\alpha_3 - \omega^2\alpha_4)\, U''\} = 0. \tag{3.29}$$

The homogeneous between the highest order nonlinear term and the linear terms of the highest order, we obtain $N = 2$. Thus, the structure of the solution of Eq. (3.29) is one and the same to the form of the solution (3.7).

Inserting the values of $U$, $U'$ and $U''$ into Eq. (3.29) and then setting each coefficient of $\psi^{-j},\ j = 0, 1, 2, \cdots$ to zero, we successively obtain

$$a_0(-1 + \omega^2 - \varepsilon a_0\alpha_1) = 0. \tag{3.30}$$

$$a_1\{(-1 + \omega^2 - 2\varepsilon a_0\alpha_1)\psi' + \varepsilon\omega\alpha_2\psi'' + \varepsilon(\alpha_3 - \omega^2\alpha_4)\psi'''\} = 0. \tag{3.31}$$

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$$-\varepsilon a_1\psi'\{(a_1\alpha_1+\omega\alpha_2)\psi'+3(\alpha_3-\omega^2\alpha_4)\psi''\}+2\varepsilon a_2\psi'\{\omega\alpha_2\psi''+(\alpha_3-\omega^2\alpha_4)\psi'''\}$$
$$+a_2[(-1+\omega^2-2\varepsilon a_0\alpha_1)(\psi')^2+2\varepsilon(\alpha_3-\omega^2\alpha_4)(\psi'')^2]=0. \quad (3.32)$$

$$-2\varepsilon a_1(a_2\alpha_1-\alpha_3+\omega^2\alpha_4)(\psi')^3-2\varepsilon a_2\{\omega\alpha_2\psi'+5(\alpha_3-\omega^2\alpha_4)\psi''\}(\psi')^2=0. \quad (3.33)$$

$$-\varepsilon a_2(a_2\alpha_1-6\alpha_3+6\omega^2\alpha_4)(\psi')^4=0. \quad (3.34)$$

From Eqs. (3.30) and (3.34), we obtain

$$a_0=0,\ \frac{-1+\omega^2}{\varepsilon\alpha_1}\ \text{ and }\ a_2=\frac{6(\alpha_3-\omega^2\alpha_4)}{\alpha_1},\quad \text{scince } a_2\neq 0.$$

Therefore, depending on the values of $a_0$, the following different cases arise:

**Case 1:** When $a_0=0$, from Eqs. (3.31) - (3.33), we get

$$\psi(\xi)=c_2+\frac{30c_1(\alpha_3-\omega^2\alpha_4)}{-5a_1\alpha_1-6\omega\alpha_2}\,e^{\frac{\xi(-5a_1\alpha_1-6\omega\alpha_2)}{30(\alpha_3-\omega^2\alpha_4)}},$$

$$a_1=0,\ \ \omega=\pm\frac{\sqrt{\frac{6\varepsilon\alpha_2^2-25(\alpha_3+\alpha_4)+\sqrt{\{6\varepsilon\alpha_2^2-25(\alpha_3+\alpha_4)\}^2-2500\alpha_3\alpha_4}}{-\alpha_4}}}{5\sqrt{2}}=\pm\theta,$$

and

$$a_1=\frac{3\left[3\varepsilon\omega\alpha_1\alpha_2+5\sqrt{\varepsilon\alpha_1^2\{\varepsilon\omega^2\alpha_2^2+4(-1+\omega^2)(-\alpha_3+\omega^2\alpha_4)\}}\right]}{5\varepsilon\alpha_1^2},$$

$$\omega=-\frac{\sqrt{25+\frac{6\varepsilon\alpha_2^2}{\alpha_4}+\frac{25\alpha_3}{\alpha_4}\pm\frac{\sqrt{(-6\varepsilon\alpha_2^2-25\alpha_3-25\alpha_4)^2-2500\alpha_3\alpha_4}}{\alpha_4}}}{5\sqrt{2}},$$

where $c_1$ and $c_2$ are integration constants.

Hence for the values of $a_1$ and $\omega$, there also arise three cases. But when $a_1\neq 0$ then the shape of the solutions for dissipative case is distorted and the solution size is very long. So we have omitted the other value of $a_1$ and discussed only for $a_1=0$.

When $a_1=0$ then we get also the solutions to the above mentioned equation depends for the values of $\omega$. Thus,

$$\psi(\xi)=c_2-\frac{5c_1(\alpha_3-\omega^2\alpha_4)}{\omega\alpha_2}\,e^{-\frac{\xi\omega\alpha_2}{5(\alpha_3-\omega^2\alpha_4)}}$$

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Now, by means of the values of $a_0$, $a_1$, $a_2$ and $\psi(\xi)$ from Eq. (3.7), we achieve the subsequent solution:

$$U(\xi) = -\frac{6\omega^2 c_1^2 \alpha_2^2(-\alpha_3+\omega^2\alpha_4)}{\alpha_1\left\{\omega c_2\alpha_2 e^{\frac{\xi\omega\alpha_2}{5\alpha_3-5\omega^2\alpha_4}} - 5c_1(\alpha_3-\omega^2\alpha_4)\right\}^2}. \tag{3.35}$$

Simplifying the required solution (3.35), we derive the following close-form solution of the strain wave equation in microstructured solids for dissipative case (3.27):

$$u(x,t) = \left[6\omega^2\{-\cosh(2\sigma(x-t\omega)) + \sinh(2\sigma(x-t\omega))\}c_1^2\alpha_2^2(-\alpha_3+\omega^2\alpha_4)\right] \Big/ \Big(\alpha_1\big[\omega\{\cosh(\sigma(x-t\omega)) + \sinh(\sigma(x-t\omega))\}c_2\alpha_2 + 5\{-\cosh(\sigma(x-t\omega)) + \sinh(\sigma(x-t\omega))\}c_1(\alpha_3-\omega^2\alpha_4)\big]^2\Big). \tag{3.36}$$

where $\sigma = \frac{\omega\,\alpha_2}{10(\alpha_3-\omega^2\alpha_4)}$, $\omega = \pm\theta$ or and $c_1$, $c_2$ are integrating constants. Since $c_1$ and $c_2$ are integration constants, one might arbitrarily select their values. If we choose $c_1 = \alpha_2\omega$ and $c_2 = -5(\alpha_3-\omega^2\alpha_4)$, then from (3.36), we obtain

$$u_9(x,\ t) = \frac{3\omega^2\alpha_2^2}{50\alpha_1(\alpha_3-\omega^2\alpha_4)}\left\{1+\tanh\left(\frac{\omega(-x+t\omega)\alpha_2}{10(\alpha_3-\omega^2\alpha_4)}\right)\right\}^2. \tag{3.37}$$

Again if we choose $c_1 = \alpha_2\omega$ and $c_2 = 5(\alpha_3-\omega^2\alpha_4)$, then from (3.36), we attain the subsequent soliton solution:

$$u_{10}(x,\ t) = \frac{3\omega^2\alpha_2^2}{50\alpha_1(\alpha_3-\omega^2\alpha_4)}\left\{1+\coth\left(\frac{\omega(-x+t\omega)\alpha_2}{10(\alpha_3-\omega^2\alpha_4)}\right)\right\}^2. \tag{3.38}$$

**Case 2:** When $a_0 = \dfrac{-1+\omega^2}{\varepsilon\,\alpha_1}$, from Eq.(3.31)-(3.33), we obtain

$$\psi(\xi) = c_2 + \frac{30c_1(\alpha_3-\omega^2\alpha_4)}{-5a_1\alpha_1-6\omega\alpha_2}\, e^{\frac{\xi(-5a_1\alpha_1-6\omega\alpha_2)}{30(\alpha_3-\omega^2\alpha_4)}},$$

where $c_1$ and $c_2$ are integration constants and

$$\left\{a_1 = 0,\ \ \omega = \left[\begin{array}{l} \pm\dfrac{\sqrt{\dfrac{6\varepsilon\alpha_2^2+25\alpha_3+25\alpha_4-\sqrt{\{6\varepsilon\alpha_2^2+25(\alpha_3+\alpha_4)\}^2-2500\alpha_3\alpha_4}}{\alpha_4}}}{5\sqrt{2}} = \pm\vartheta_1(\text{say}) \\ \pm\dfrac{\sqrt{\dfrac{6\varepsilon\alpha_2^2+25\alpha_3+25\alpha_4+\sqrt{\{6\varepsilon\alpha_2^2+25(\alpha_3+\alpha_4)\}^2-2500\alpha_3\alpha_4}}{\alpha_4}}}{5\sqrt{2}} = \pm\vartheta_2(\text{say}) \end{array}\right.\right\};$$

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$$\left\{ a_1 = \frac{3\left[3\varepsilon\omega\alpha_1\alpha_2 + 5\sqrt{\varepsilon\alpha_1^2\{\varepsilon\omega^2\alpha_2^2 + 4(-1+\omega^2)(\alpha_3 - \omega^2\alpha_4)\}}\right]}{5\varepsilon\alpha_1^2}, \right.$$

$$\left. \omega = -\frac{\sqrt{\frac{-6\varepsilon\alpha_2^2 + 25\alpha_3 + 25\alpha_4 \pm \sqrt{\{6\varepsilon\alpha_2^2 - 25(\alpha_3+\alpha_4)\}^2 - 2500\alpha_3\alpha_4}}{\alpha_4}}}{5\sqrt{2}} \right\};$$

$$\left\{ a_1 = \frac{3\left[3\varepsilon\omega\alpha_1\alpha_2 - 5\sqrt{\varepsilon\alpha_1^2\{\varepsilon\omega^2\alpha_2^2 + 4(-1+\omega^2)(\alpha_3 - \omega^2\alpha_4)\}}\right]}{5\varepsilon\alpha_1^2}, \right.$$

$$\left. \omega = \frac{\sqrt{\frac{-6\varepsilon\alpha_2^2 + 25\alpha_3 + 25\alpha_4 \pm \sqrt{\{6\varepsilon\alpha_2^2 - 25(\alpha_3+\alpha_4)\}^2 - 2500\alpha_3\alpha_4}}{\alpha_4}}}{5\sqrt{2}} \right\}.$$

Hence for the values of $a_1$ and $\omega$, there arises also three cases. When $a_1 \neq 0$, then the form of solutions to the strain wave equation in microstructured solids for dissipative case (3.24) indistinct and the solution size is very lengthy. So we omitted the extra value of $a_1$ and only discuss for $a_1 = 0$.

When $a_1 = 0$ then we find also the solutions to the above revealed equation depends for the values of $\omega$, i.e. $\omega = \pm\vartheta_1$ and $\omega = \pm\vartheta_2$. Therefore,

$$\psi(\xi) = c_2 - \frac{5c_1(\alpha_3 - \omega^2\alpha_4)}{\omega\alpha_2}\, e^{-\frac{\xi\omega\alpha_2}{5(\alpha_3 - \omega^2\alpha_4)}}$$

where . $\omega = \pm\vartheta_1$ or $\omega = \pm\vartheta_2$, $c_1$ and $c_2$ are constants of integration.

Substituting the values of $a_0$, $a_1$, $a_2$ and $\psi(\xi)$ into Eq. (3.7), we accomplish the following solution:

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$$U(\xi) = \frac{-1+\omega^2}{\varepsilon\alpha_1} - \frac{6\omega^2 c_1^2 \alpha_2^2(-\alpha_3+\omega^2\alpha_4)}{\alpha_1\left\{\omega c_2\alpha_2 e^{\frac{\xi\omega\alpha_2}{5\alpha_3-5\omega^2\alpha_4}} - 5c_1(\alpha_3-\omega^2\alpha_4)\right\}^2}. \quad (3.39)$$

Simplifying the required exponential function solution (3.39) into trigonometric function solution, we derive the solution of Eq. (3.27) as follows:

$$\begin{aligned} u(x,t) = {} & \left[\omega^2(-1+\omega^2)\{\cosh(2\varphi(x-t\omega))+\sinh(2\varphi(x-t\omega))\}c_2^2\alpha_2^2\right. \\ & + \{\cosh(2\varphi(x-t\omega))-\sinh(2\varphi(x-t\omega))\}c_1^2(\alpha_3-\omega^2\alpha_4)\{6\varepsilon\omega^2\alpha_2^2 \\ & \left. -25(-1+\omega^2)(-\alpha_3+\omega^2\alpha_4)\} + 10\omega(-1+\omega^2)c_1c_2\alpha_2(-\alpha_3+\omega^2\alpha_4)\right] \\ & \Big/ \Big(\varepsilon\alpha_1\big[\omega\{\cosh(\varphi(x-t\omega))+\sinh(\varphi(x-t\omega))\}c_2\alpha_2 \\ & + 5\{-\cosh(\varphi(x-t\omega))+\sinh(\varphi(x-t\omega))\}c_1(\alpha_3-\omega^2\alpha_4)\big]^2\Big). \quad (3.40) \end{aligned}$$

Therefore, we obtain the generalized soliton solution (3.40) to the strain wave equation in microstructured solids for dissipative case, where $\varphi = \frac{\omega\,\alpha_2}{10(\alpha_3-\omega^2\alpha_4)}$ and . $\omega = \pm\vartheta_1$ or $\omega = \pm\vartheta_2$. But, since $c_1$ and $c_2$ are arbitrary constants, someone may arbitrarily choose their values. So, if we choose $c_1 = \alpha_2\omega$ and $c_2 = 5(\alpha_3-\omega^2\alpha_4)$, from (3.20), we get the subsequent soliton solutions:

$$u_{11}(x,\ t) = \frac{(-1+\omega^2)}{\alpha_1\varepsilon} - \frac{3\omega^2\alpha_2^2}{50\alpha_1(-\alpha_3+\omega^2\alpha_4)}\left\{-1+\coth\left(\frac{\omega(x-t\omega)\alpha_2}{10(\alpha_3-\omega^2\alpha_4)}\right)\right\}^2. \quad (3.41)$$

Again, if we choose $c_1 = \alpha_2\omega$ and $c_2 = -5(\alpha_3-\omega^2\alpha_4)$, the solitary wave solution (3.40) becomes

$$u_{12}(x,\ t) = \frac{(-1+\omega^2)}{\varepsilon\alpha_1} + \frac{3\varepsilon\omega^2\alpha_2^2}{50\varepsilon\alpha_1(\alpha_3-\omega^2\alpha_4)}\left\{-1+\tanh\left(\frac{\omega(x-t\omega)\alpha_2}{10(\alpha_3-\omega^2\alpha_4)}\right)\right\}^2. \quad (3.42)$$

As $c_1$ and $c_2$ are arbitrary constants, one may pick many other values of them and each of this selection construct new solution. But for minimalism, we have not recorded these solutions.

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**Remark 2**: The solutions (3.37)-(3.38), where $\omega = \pm\theta_1$ or $\omega = \pm\theta_2$ and the solutions (3.41)-(3.42) $\omega = \pm\vartheta_1$ or $\omega = \pm\vartheta_2$ have been confirmed by satisfying the original equation.

## 4. PHYSICAL INTERPRETATIONS OF THE SOLUTIONS

In this sub-section, we draw the graph of the derived solutions and explain the effect of the parameters on the solutions for both non-dissipative and dissipative cases. The solution $u_1$ in (3.17) depends on the physical parameters $\alpha_1, \alpha_3, \alpha_4, \varepsilon$ and the group velocity $\omega$ . Now, we will discuss all the possible physical significances for $-2 \le \alpha_1, \alpha_3, \alpha_4, \varepsilon \le 2$, and soliton exists for $|\omega| > 1$ and $|\omega| < 1$. For the value of parameters $\alpha_1, \alpha_3, \alpha_4, \varepsilon < 0$ and $|\omega| > 1$, the solution $u_1$ in (3.17) represents the bell shape soliton and when $|\omega| < 1$ then the solution $u_1$ represents the anti-bell shape soliton. It is shown in Fig. 1. Also if the values of the parameters are $\alpha_1 > 0,\ \alpha_3, \alpha_4, \varepsilon < 0$ and $|\omega| > 1$, then the solution $u_1$ represents the anti-bell shape soliton and when $|\omega| < 1$, then the solution $u_1$ represents the bell shape soliton. It is shown the Fig. 2. Again, for $\alpha_1, \alpha_3, \alpha_4 < 0, \varepsilon > 0$ and $|\omega| < 1$, the solution $u_1$ in (3.17) represents the multi-soliton and when $|\omega| > 1$, the solution $u_1$ represents the anti-bell shape soliton. It is plotted in Fig. 3. Again, if the values of the physical parameters are $\alpha_1 > 0,\ \alpha_3, \alpha_4 < 0, \varepsilon > 0$ and $|\omega| > 1$, then the solution $u_1$ represents the anti-bell shape soliton and when $|\omega| < 1$ then the solution $u_1$ represents the bell shape soliton. It is shown in Fig. 4. We can sketch the other figures of the solution $u_1$ for different values of the parameters. But for page limitation in this article we have omitted these figures. So, for other cases we do not draw the figures but we discuss for other cases with the following table:

| | | | |
|---|---|---|---|
| $\varepsilon > 0$ | $\lvert\omega\rvert > 1$ | $\alpha_1 < 0, \alpha_3 < 0, \alpha_4 < 0$ | Anti-bell shape soliton |
| | | $\alpha_1 > 0, \alpha_3 < 0, \alpha_4 < 0$ | Bell shape soliton |
| | | $\alpha_1 > 0, \alpha_3 > 0, \alpha_4 < 0$ | Bell shape soliton |
| | | $\alpha_1 > 0, \alpha_3 > 0, \alpha_4 > 0$ | Bell shape soliton |
| | | $\alpha_1 > 0, \alpha_3 < 0, \alpha_4 > 0$ | Bell shape soliton |
| | | $\alpha_1 < 0, \alpha_3 > 0, \alpha_4 < 0$ | Anti-bell shape soliton |

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| | | | |
|---|---|---|---|
| | | $\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 > 0$ | Anti-bell shape soliton |
| | | $\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 > 0$ | Anti-bell shape soliton |
| | $\lvert\omega\rvert < 1$ | $\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 < 0$ | Bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 < 0$ | Anti-bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 < 0$ | Anti-bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 > 0$ | Anti-bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 > 0$ | Anti-bell shape soliton |
| | | $\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 < 0$ | Bell shape soliton |
| | | $\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 > 0$ | Bell shape soliton |
| | | $\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 > 0$ | Periodic bell shape soliton |
| $\varepsilon < 0$ | $\lvert\omega\rvert > 1$ | $\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 < 0$ | Bell shape or Periodic bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 < 0$ | Anti-bell shape soliton or Periodic anti-bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 < 0$ | Anti-bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 > 0$ | Periodic anti-bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 > 0$ | Periodic anti-bell shape soliton |
| | | $\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 < 0$ | Bell shape soliton |
| | | $\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 > 0$ | Periodic bell shape soliton |
| | | $\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 > 0$ | Periodic bell shape soliton |
| | $\lvert\omega\rvert < 1$ | $\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 < 0$ | Anti-bell shape soliton or Periodic anti-bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 < 0$ | Bell shape or Periodic bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 < 0$ | Periodic bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 > 0,\ \alpha_4 > 0$ | Bell shape or Periodic bell shape soliton |
| | | $\alpha_1 > 0,\ \alpha_3 < 0,\ \alpha_4 > 0$ | Bell shape soliton |
| | | $\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 < 0$ | Periodic anti-bell shape soliton |
| | | $\alpha_1 < 0,\ \alpha_3 > 0,\ \alpha_4 > 0$ | Anti-bell shape soliton or Periodic anti-bell shape soliton |
| | | $\alpha_1 < 0,\ \alpha_3 < 0,\ \alpha_4 > 0$ | Anti-bell shape soliton |

Also the soliton $u_2$ in (3.18) depends on the parameters $\alpha_1, \alpha_3, \alpha_4, \varepsilon$ and $\omega$. Now, we will discus all the possible physical significances for $-2 \le \alpha_1, \alpha_3, \alpha_4, \varepsilon \le 2$, and soliton exists for $\lvert\omega\rvert > 1$ and $\lvert\omega\rvert < 1$. For the value of parameters contains $\alpha_1, \alpha_3, \alpha_4, \varepsilon > 0$ and $\lvert\omega\rvert > 1$, then the solution $u_2$ in (3.18) represents the singular anti-bell shape soliton and when $\lvert\omega\rvert < 1$ then the solution $u_2$ represents the singular bell shape soliton. It is shown in Fig. 5. Also, for $\alpha_1, \alpha_3, \alpha_4 < 0, \varepsilon > 0$ and $\lvert\omega\rvert > 1$, then the solution $u_2$ in (3.18) represents the periodic

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singular anti-bell shape soliton and when $|\omega|<1$ then the solution $u_2$ represents the periodic singular bell shape soliton. It is plotted of the Fig. 6. On the other hand, the solutions $u_3$ in (3.19) and $u_4$ in (3.20) exist for $(\alpha_3-\alpha_4\,\omega^2)>0,\ \varepsilon<0$ or $(\alpha_3-\alpha_4\,\omega^2)<0,\ \varepsilon>0$ when $|\omega|>1$ or $|\omega|>1$. For the value of the parameters are $\alpha_1=-1.25,\ \alpha_3=-0.1,\ \alpha_4=-2,\ \varepsilon=-1$, when $\omega=0.96$, the solution $u_3$ in (3.19) represents the anti-bell shape soliton and $\alpha_1=-1.5,\ \alpha_3=-0.1,\ \ \alpha_4=2,\ \varepsilon=-1$, when $\omega=1.5$, the solution $u_4$ represents the periodic soliton. It is shown in Fig. 7. Again, the travelling wave solution $u_5$ in (3.23) represents the bell shape singular solitons for $\alpha_1=-1=\alpha_3,\ \ \alpha_4=1,\ \varepsilon=0.5,\ \ \omega=-1.5$ and $\omega=0.5$ respectively, in Fig. 8 and Fig. 9 from $u_6$ in (3.24) represents the bell shape soliton, when $\omega=1.5$ and the anti-bell shape soliton, when $\omega=-0.75$. In Fig. 10, we have plotted of the periodic bell shape and anti-bell shape soliton for $\alpha_1=\alpha_3=-1.25$, $\alpha_4=1,\ \ \varepsilon=0.7$, $\omega=-1.2$ and $\alpha_1=\alpha_3=-1.25$, $\alpha_4=1,\ \ \varepsilon=-0.7$, $\omega=0.25$ respectively to the solution of $u_7$ in (3.25) and Fig. 11 plotted the periodic anti-bell shape soliton and bell shape soliton for $\alpha_1=1.25,\ \ \alpha_3=-1.25$, $\alpha_4=1,\ \ \varepsilon=0.7$, $\omega=-1.2$ and $\alpha_1=\alpha_3=-1.25$, $\alpha_4=1,\ \ \varepsilon=-0.7$, $\omega=-0.25$ respectively to the solution of $u_8$ in (3.26). Fig. 12 and 13 represent the kink shape solutions $u_9$ given in (3.37) are respectively, for $\alpha_1=1,\ \ \alpha_2=1,\ \ \alpha_3=-1.5,\ \ \alpha_4=-1$ and $\alpha_1=-1,\ \ \alpha_2=1,\ \ \alpha_3=-1.5,\ \ \alpha_4=-1$ respectively, when $\omega=\pm\mu_1$ and for $\alpha_1=1,\ \alpha_2=1,\ \ \alpha_3=-1.5,\ \ \alpha_4=-1$ and $\alpha_1=-1,\ \ \alpha_2=1,\ \ \alpha_3=-1.5,\ \ \alpha_4=-1$ respectively, when $\omega=\pm\mu_2$. Also sketch the figures 14 and 15, singular bell shape solutions $u_{10}$ in (3.38) for $\alpha_1=1,\ \ \alpha_2=1,\ \ \alpha_3=-1.5,\ \ \alpha_4=-1$ and $\alpha_1=-1,\ \ \alpha_2=1,\ \ \alpha_3=-1.5,\ \ \alpha_4=-1$ respectively, when $\omega=\pm\mu_1$ and for $\alpha_1=1,\ \ \alpha_2=1,\ \ \alpha_3=-1.5,\ \ \alpha_4=-1$ and $\alpha_1=-1,\ \ \alpha_2=1,\ \ \alpha_3=-1.5,\ \alpha_4=-1$ respectively, when $\omega=\pm\mu_2$. On the other hand, Fig. 16 and 17 are singular bell and singular anti-bell shape soliton solitons $u_{11}$ in (3.41) for $\alpha_1=1,\ \ \alpha_2=1,\ \ \alpha_3=1,\ \ \alpha_4=1,\ \varepsilon=0.5$ and $\alpha_1=-1,\ \ \alpha_2=1,\ \ \alpha_3=1,\ \ \alpha_4=1,\ \ \varepsilon=0.5$ respectively, when $\omega=\pm\theta_1$ and for

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$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 1, \quad \alpha_4 = 1, \ \varepsilon = 0.5$ and $\alpha_1 = -1, \quad \alpha_2 = 1, \quad \alpha_3 = 1, \quad \alpha_4 = 1, \ \varepsilon = 0.5$ respectively, when $\omega = \pm\theta_2$. Also, draw the Figures 18 and 19 are kink shape solitons $u_{12}$ in (3.42) for $\alpha_1 = 1, \ \alpha_2 = 1, \ \alpha_3 = 1, \ \alpha_4 = 1, \ \varepsilon = 0.5$ and $\alpha_1 = -1, \ \alpha_2 = 1, \ \alpha_3 = 1, \ \alpha_4 = 1, \ \varepsilon = 0.5$ respectively, when $\omega = \pm\theta_1$ and for $\alpha_1 = 1, \ \alpha_2 = 1, \ \alpha_3 = 1, \ \alpha_4 = 1, \ \varepsilon = 0.5$ and $\alpha_1 = -1, \ \alpha_2 = 1, \ \alpha_3 = 1, \ \alpha_4 = 1, \ \varepsilon = 0.5$ respectively, when $\omega = \pm\theta_2$. All figures are drawn within $-10 \le x, \ t \le 10$.

We can sketch the other figures or discuss the solutions $u_2$ to $u_{12}$ for different values of the parameters. But for page limitation in this article we have omitted these figures in details.

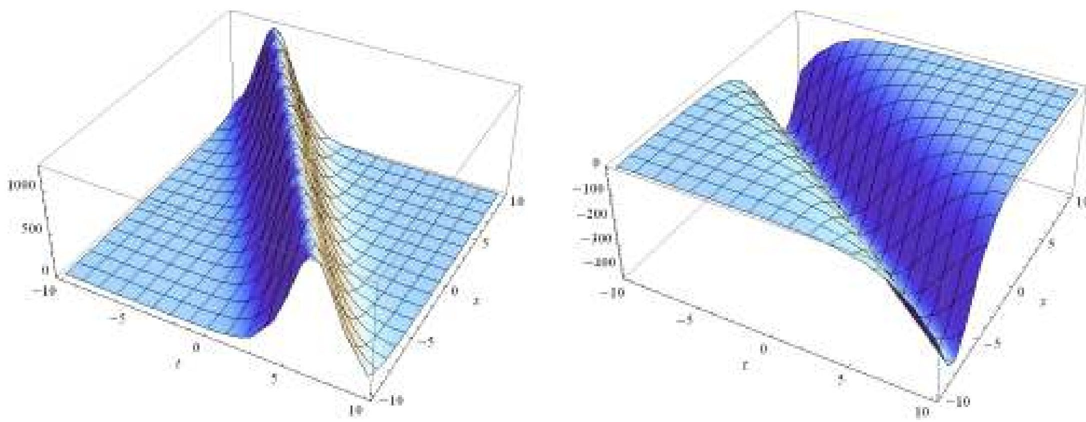

Fig. 1: Sketch of the solution $u_1$ for $\alpha_1 = -0.001, \ \alpha_3 = \alpha_4 = \varepsilon = \omega = -1.5$ and $\alpha_1 = -0.001, \ \alpha_3 = \alpha_4 = \varepsilon = \omega = -0.75$ respectively.

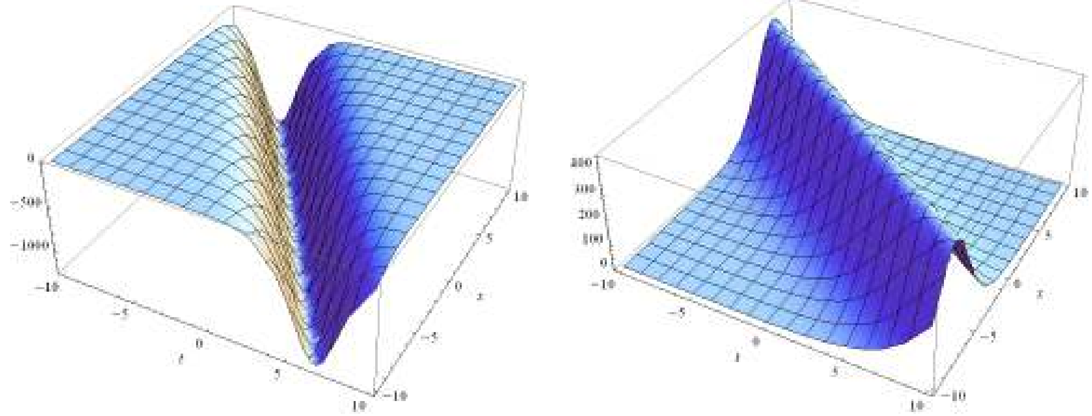

Fig. 2: Plot of the solution $u_1$ for $\alpha_1 = 0.001, \ \alpha_3 = \alpha_4 = \varepsilon = \omega = -1.5$ and $\alpha_1 = 0.001, \ \alpha_3 = \alpha_4 = \varepsilon = \omega = -0.75$ respectively.

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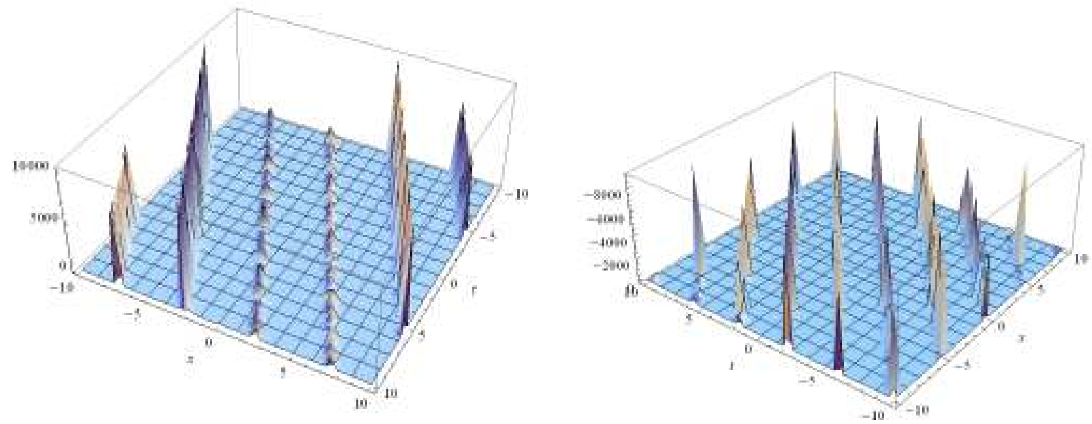

Fig. 3: Sketch of the solution $u_1$ for $\alpha_1 = \alpha_3 = \alpha_4 = -1.2,\ \varepsilon = \omega = 0.5$ and $\alpha_1 = \alpha_3 = \alpha_4 = -1.2,\ \varepsilon = 0.5,\ \omega = 1.25$ respectively.

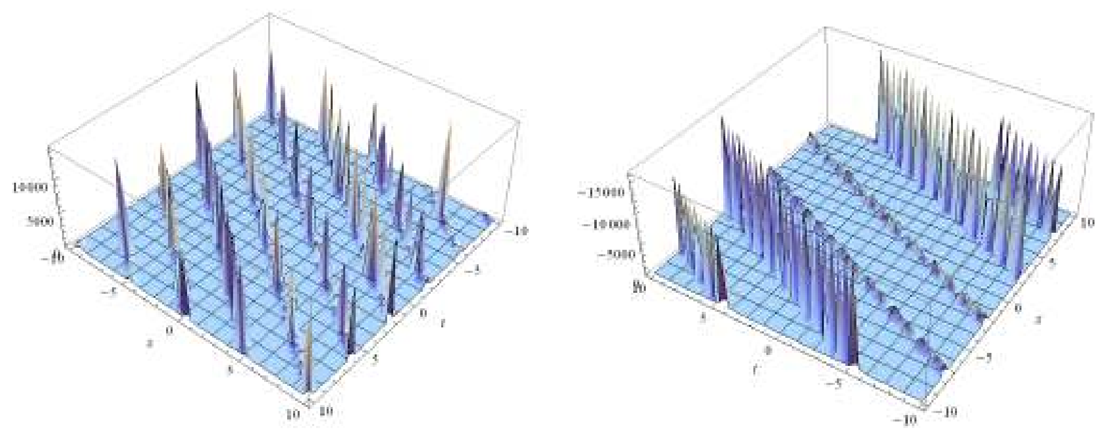

Fig. 4: Sketch of the solution $u_1$ for $\alpha_1 = 0.75,\ \alpha_3 = \alpha_4 = -1.2$, $\varepsilon = 0.5$, $\omega = 1.25$ and $\alpha_1 = 0.75,\ \alpha_3 = \alpha_4 = -1.2,\ \varepsilon = 0.5,\ \omega = 0.5$ respectively.

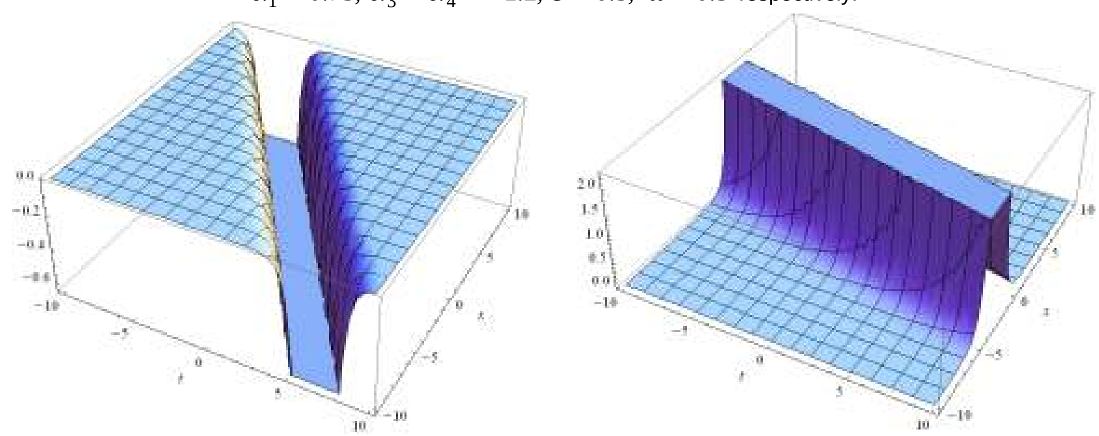

Fig. 5: Sketch of the singular dark and singular bell shape soliton $u_2$ for $\alpha_1 = \alpha_3 = \alpha_4 = 0.5$, $\varepsilon = 0.75$, $\omega = -1.5$ and $\alpha_1 = \alpha_3 = \alpha_4 = 0.5$, $\varepsilon = 0.75$, $\omega = -0.25$ respectively.

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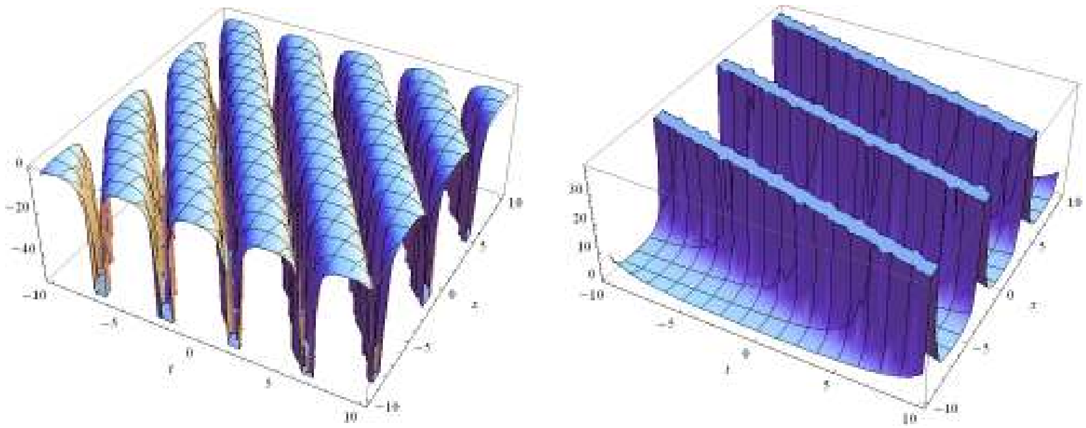

Fig. 6: Sketch of the periodic singular solution $u_2$ for $\alpha_1 = \alpha_3 = \alpha_4 = -1.5$, $\varepsilon = 0.75$, $\omega = -1.5$ and $\alpha_1 = \alpha_3 = \alpha_4 = -1.5$, $\varepsilon = 0.75$, $\omega = -0.25$ respectively.

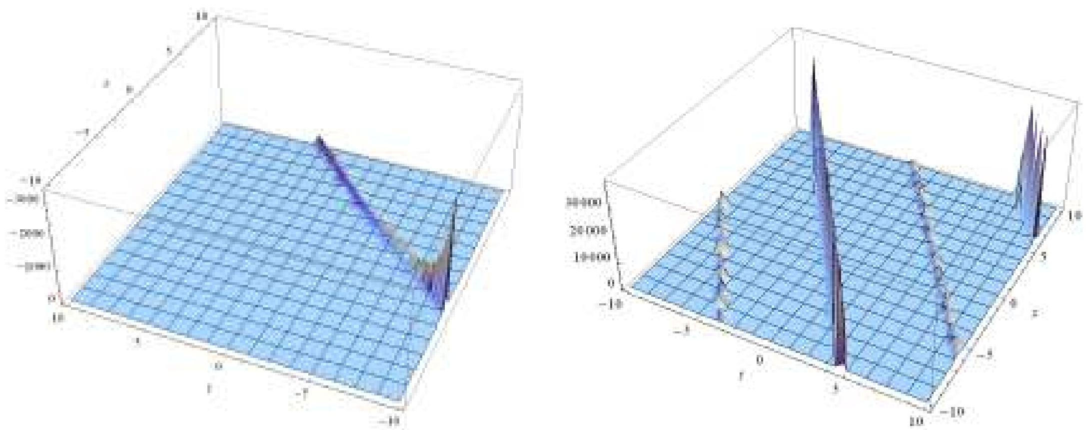

Fig. 7: Sketch of the solution $u_3$ and the solution $u_4$ for $\alpha_1 = -1.25$, $\alpha_3 = -0.1$, $\alpha_4 = -2$, $\varepsilon = -1$, $\omega = 0.96$ and $\alpha_1 = -1.5$, $\alpha_3 = -0.1$, $\alpha_4 = 2$, $\varepsilon = -1$, $\omega = 1.5$ respectively.

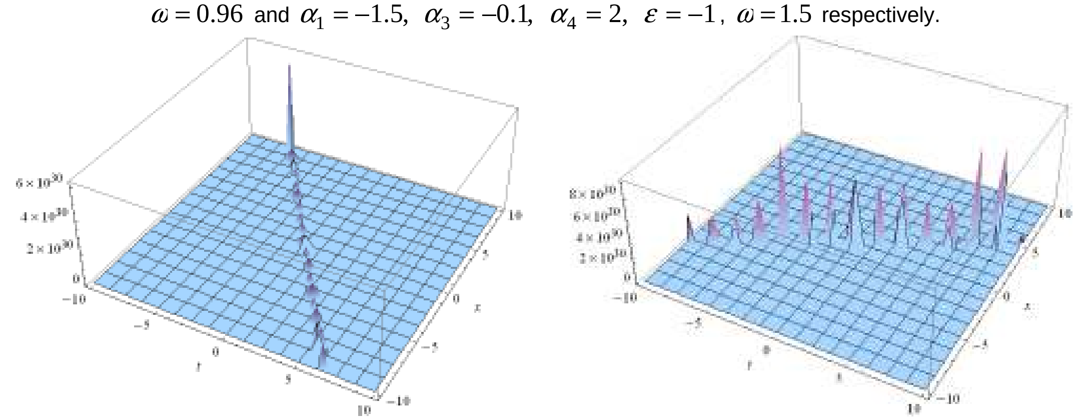

Fig. 8: Sketch of the solutions $u_5$ for $\alpha_1 = -1 = \alpha_3$, $\alpha_4 = 1$, $\varepsilon = 0.5$, $\omega = -1.5$ and $\omega = 0.5$ respectively.

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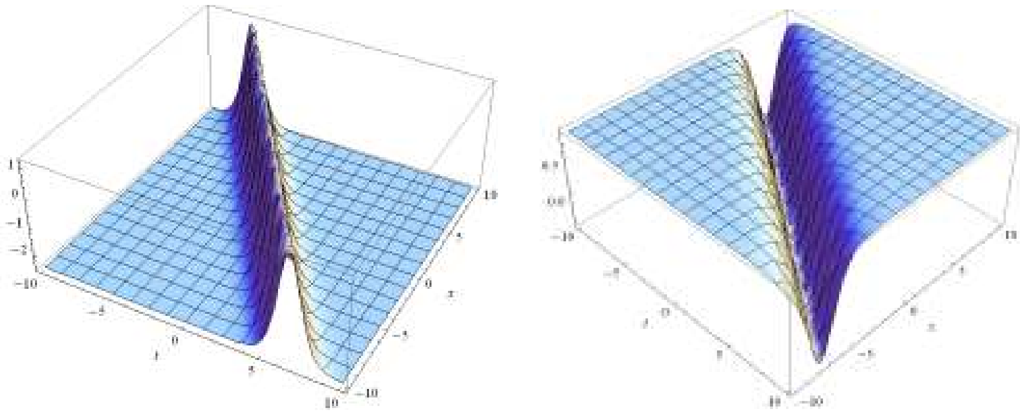

Fig. 9: Sketch of the bell shape soliton and anti-bell shape soliton $u_6$ for $\alpha_1 = \alpha_3 = \alpha_4 = -1$, $\varepsilon = 0.5$, $\omega = 1.5$ and $\omega = -0.75$ respectively.

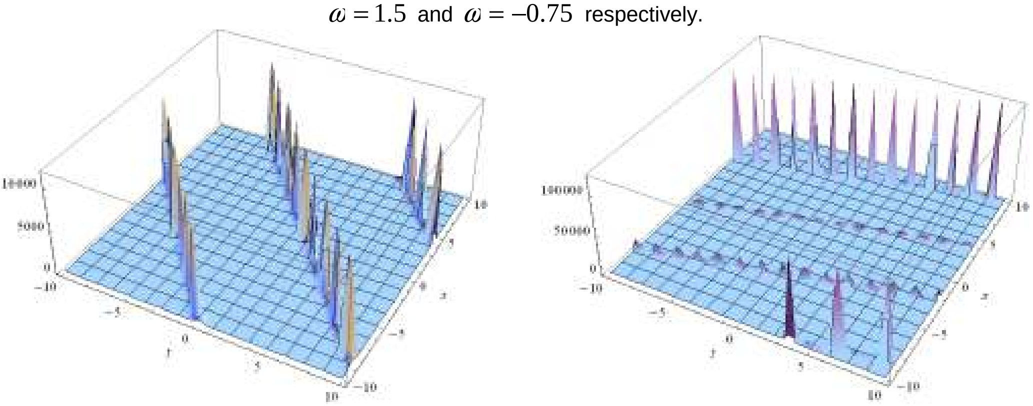

Fig. 10: Sketch of the solutions $u_7$ for $\alpha_1 = \alpha_3 = -1.25$ , $\alpha_4 = 1$, $\varepsilon = 0.7$, $\omega = -1.2$ and $\alpha_1 = \alpha_3 = -1.25$ , $\alpha_4 = 1$, $\varepsilon = -0.7$ , $\omega = 0.25$ respectively.

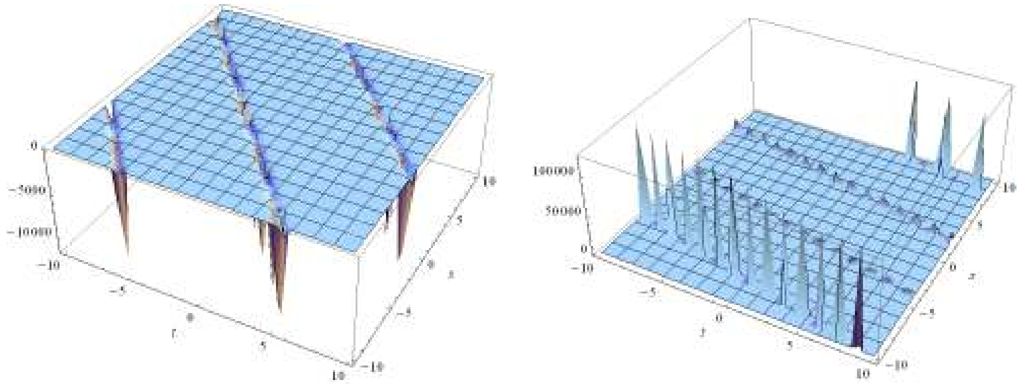

Fig. 11: Sketch of the solutions $u_8$ for $\alpha_1 = 1.25$, $\alpha_3 = -1.25$ , $\alpha_4 = 1$, $\varepsilon = 0.7$, $\omega = -1.2$ and $\alpha_1 = \alpha_3 = -1.25$ , $\alpha_4 = 1$, $\varepsilon = -0.7$ , $\omega = -0.25$ respectively.

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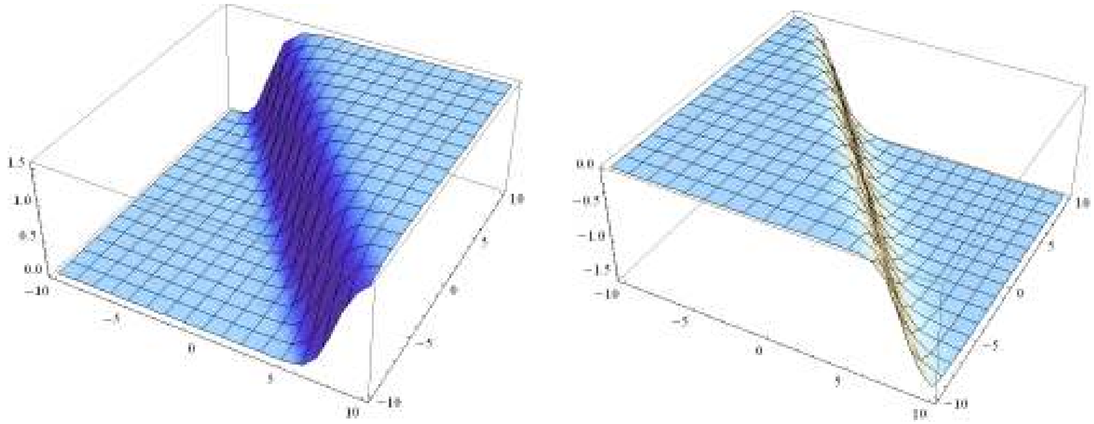

Fig. 12: Kink shape soliton obtained from $u_9$ for $\alpha_1 = 1, \quad \alpha_2 = 1, \ \alpha_3 = -1.5, \quad \alpha_4 = -1, \ \varepsilon = 0.5$ and $\alpha_1 = -1, \quad \alpha_2 = 1, \ \alpha_3 = -1.5, \quad \alpha_4 = -1, \ \varepsilon = 0.5$ respectively, when $\omega = \pm\mu_1$.

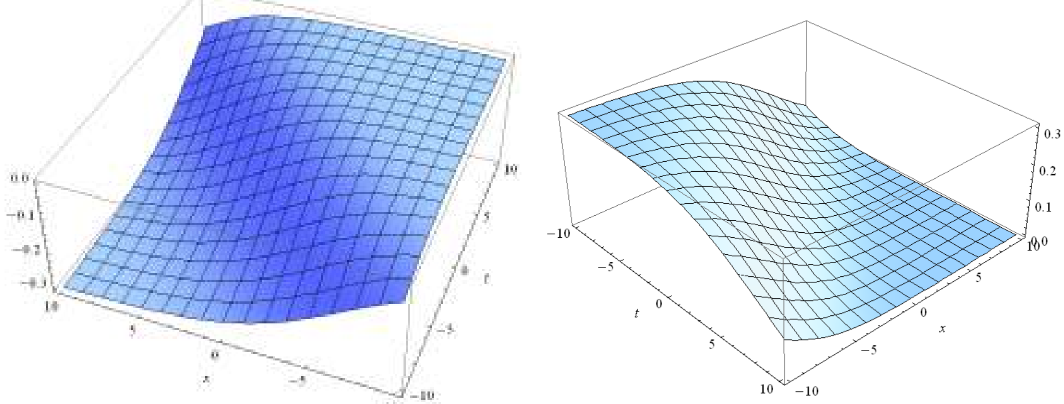

Fig. 13: Kink shape soliton obtained from $u_9$ for $\alpha_1 = 1, \quad \alpha_2 = 1, \ \alpha_3 = -1.5, \quad \alpha_4 = -1, \ \varepsilon = 0.5$ and $\alpha_1 = -1, \quad \alpha_2 = 1, \ \alpha_3 = -1.5, \quad \alpha_4 = -1, \ \varepsilon = 0.5$ respectively, when $\omega = \pm\mu_2$.

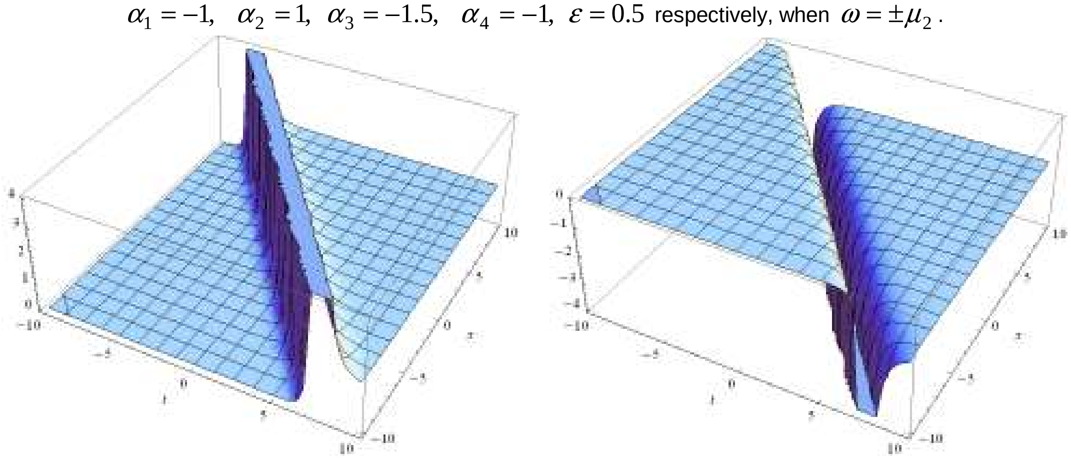

Fig. 14: Singular bell shape and anti-bell shape soliton $u_{10}$ for $\alpha_1 = 1, \quad \alpha_2 = 1, \ \alpha_3 = -1.5, \quad \alpha_4 = -1,$ $\varepsilon = 0.5$ and $\alpha_1 = -1, \quad \alpha_2 = 1, \ \alpha_3 = -1.5, \quad \alpha_4 = -1, \ \varepsilon = 0.5$ respectively, when $\omega = \pm\mu_1$.

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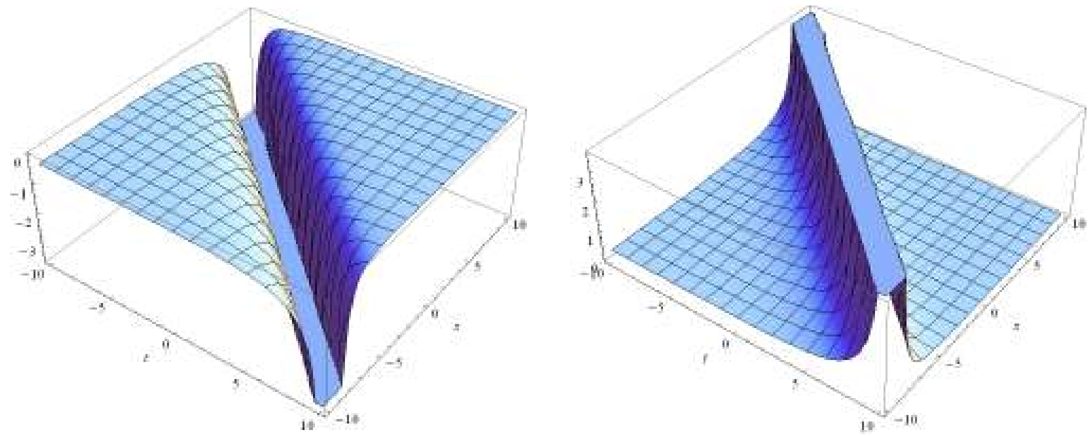

Fig. 15: Singular anti-bell shape and bell shape soliton $u_{10}$ in (3.38) for $\alpha_1 = 1, \quad \alpha_2 = 1, \ \alpha_3 = -1.5,$ $\alpha_4 = -1, \ \varepsilon = 0.5$ and $\alpha_1 = -1, \quad \alpha_2 = 1, \ \alpha_3 = -1.5, \quad \alpha_4 = -1, \ \varepsilon = 0.5$ respectively, when $\omega = \pm \mu_2$.

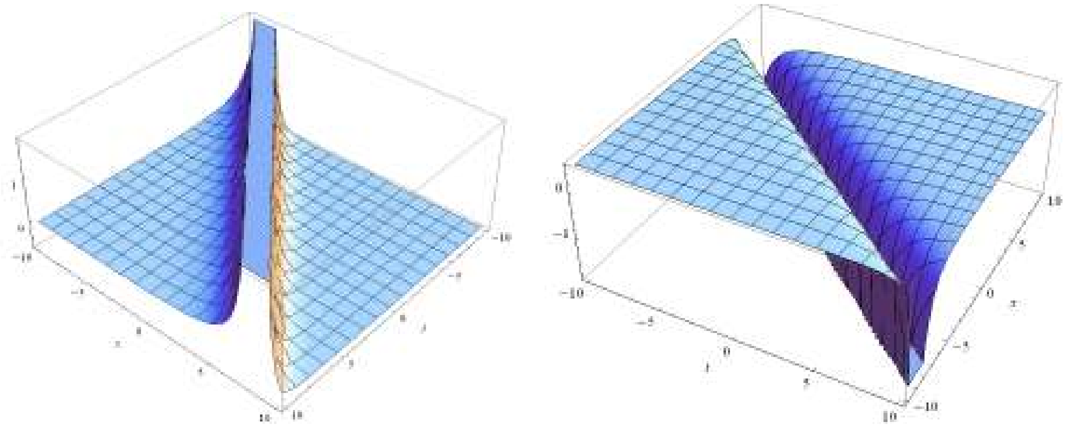

Fig. 16: Sketch the singular bell type and anti-bell soliton $u_{11}$ for $\alpha_1 = 1, \quad \alpha_2 = 1, \ \alpha_3 = 1, \ \alpha_4 = 1, \ \varepsilon = 0.5$ and $\alpha_1 = -1, \quad \alpha_2 = 1, \ \alpha_3 = 1, \ \alpha_4 = 1, \ \varepsilon = 0.5$ respectively, when $\omega = \pm \theta_1$.

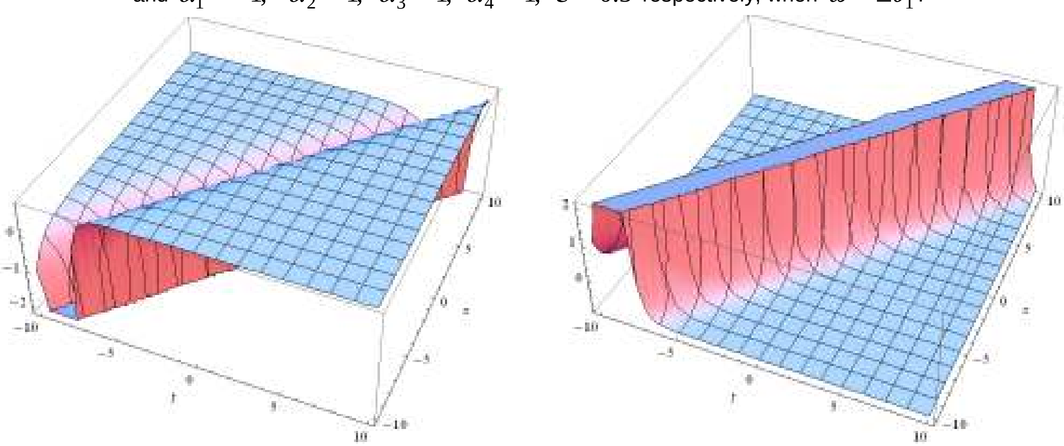

Fig. 17: Singular anti-bell shape and bell shape soliton $u_{11}$ for $\alpha_1 = 1, \quad \alpha_2 = 1, \ \alpha_3 = 1, \ \alpha_4 = 1, \ \varepsilon = 0.5$ and $\alpha_1 = -1, \quad \alpha_2 = 1, \ \alpha_3 = 1, \ \alpha_4 = 1, \ \varepsilon = 0.5$ respectively, when $\omega = \pm \theta_2$.

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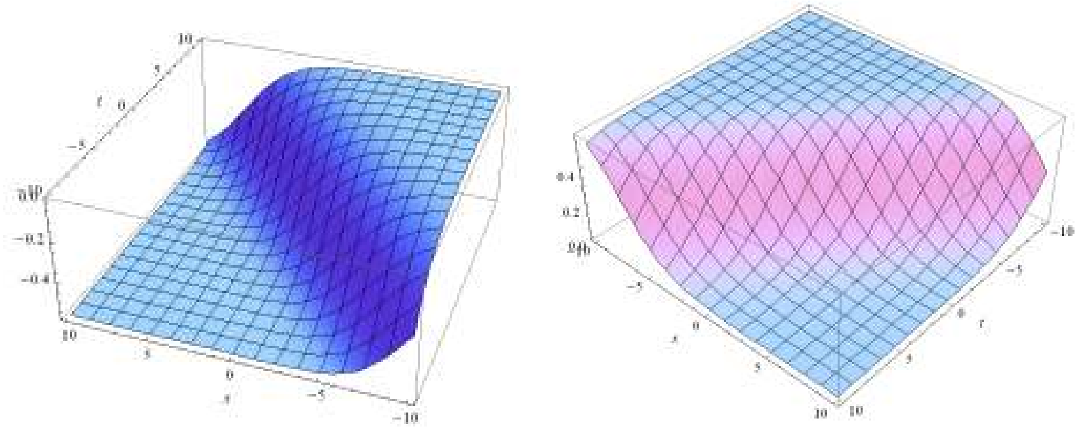

Fig. 18: Kink shape soliton $u_{12}$ for $\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 1, \quad \alpha_4 = 1, \quad \varepsilon = 0.5$ and $\alpha_1 = -1, \quad \alpha_2 = 1,$ $\alpha_3 = 1, \quad \alpha_4 = 1, \quad \varepsilon = 0.5$ respectively, when $\omega = \pm\theta_1$.

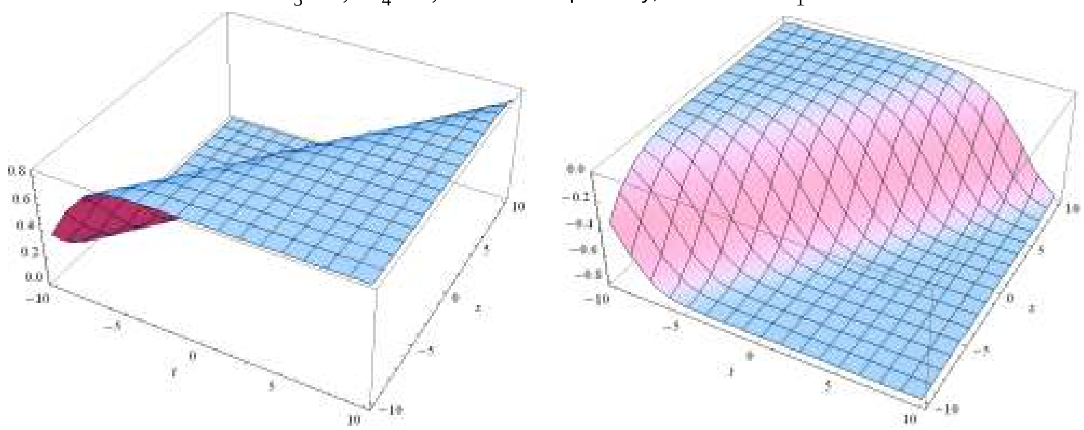

Fig. 19: Kink shape soliton $u_{12}$ for $\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 1, \quad \alpha_4 = 1, \quad \varepsilon = 0.5$ and $\alpha_1 = -1,$ $\alpha_2 = 1, \quad \alpha_3 = 1, \quad \alpha_4 = 1, \quad \varepsilon = 0.5$ respectively, when $\omega = \pm\theta_2$.

## 5. CONCLUSION

In this article, we have implemented the MSE method to obtain soliton solutions to the strain wave equation in microstructured solids for both non-dissipative and dissipative cases. In fact, we have derived general solitary wave solutions to this equation associated with arbitrary constants, and for particular values of these constants solitons are originated from the general solitary wave solutions. We have illustrated the solitary wave properties of the solutions for various values of the free parameters by means of the graphs. This work shows

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that the MSE method is competent and more powerful and can be used for many other equations NLEEs applied mathematics and engineering.

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